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A MATHEMATICAL MODEL FOR COMPUTING THE TRAJECTORIES OF ROCKETS IN A RESISTANT MEDIUM TAKING INTO ACCOUNT THE EARTH'S ROTATION. THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE ROCKET'S TRAJECTORY

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Abstract: *Within this paper, there is established the system of ordinary equations of the trajectory of a rocket with respect to the Earth, while taking into account, the aerodynamic drag, the weight, and the Earth's rotation and curvature. The local latitude and the longitude during the, flight, which implicitly appear in the expressions of the forces acting upon the rocket, are also calculated. The mathematical model thus obtained is rendered in forms utilized in ballistics. This system of equations can also he used for the calculation of the trajectory of an active-reactive projectile, and also, with some adjustments, for the calculation of the orbit of an aerospace vehicle.*

Keywords: *rocket, Earth's rotation, trajectory, rocket latitude and longitude.*

1. THE EXPRESSIONS OF THE FORCES ACTING ON THE ROCKET WITHIN THE HYPOTHESIS OF THE FUNDAMENTAL PROBLEM OF THE EXTERNAL BALLISTICS

The vector equation of the rocket's, equation $m\cdot\frac{d\overline{V}}{dt}=\overline{\mathcal{T}}+\overline{R}+\overline{\mathcal{G}}+\overline{F}_c$ of [9], contains the forces which effectively act on the rocket:

- engine's thrust, $\overline{\mathcal{T}}$;
- the aerodynamic resultant force, $\overline{\mathcal{F}}_a$, which, within the main hypothesis of the fundamental problem of ballistics, reduces to the aerodynamic drag, $\overline{R}$;
- the weight, $\overline{\mathcal{G}}$;
- the Coriolis force, $\overline{F}_c$.

1.1 FORCES EFFECTIVELY APPLIED

1.1.1 THE THRUST

It is assumed that the thrust, $\overline{\mathcal{T}}$, is oriented along the axis of the rocket, which in turn (whithin the main hypothesis of the fundamental problem of the external ballistics) is tangent to the trajectory (***Figure 1***).

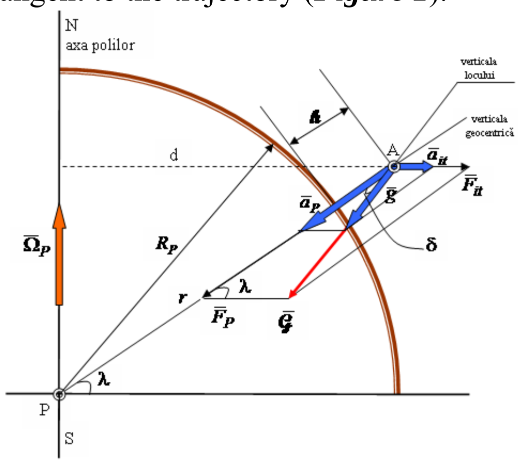

Figure 1. *Forces effectively applied*

Within the Earth-linked frame O_1xyz (the trajectory is expressed with respect to ***Figure 1***), the velocity vector $\overline{V}$ can be written as $\overline{V} = V_x \bar{i} + V_y \bar{j} + V_z \bar{k}$ **(1)**

So the thrust vector is $\overline{\mathcal{T}} = \mathcal{T} \frac{\overline{V}}{V}$ **(2)**

In general, the absolute value of the thrust is function of the flight's altitude, $\mathcal{h}$, and time [6, 11].

The absolute value of the thrust created by the rocket engine can be expressed as [5, 6] $\mathcal{T} = \mathcal{T}_0(t) + S_e p_0 [1 - \mathcal{B}(\mathcal{h})]$, **(3)** where $\mathcal{T}_0(t)$ is the ground-level thrust (which depends upon the time t), S_e - the area of the nozzle's exit section, p_0 - the ground-level air pressure, and $\mathcal{B}(\mathcal{h})$ - the function which indicates the relative variation of the air pressure with respect to the altitude, $\mathcal{h}$, $\mathcal{B}(\mathcal{h}) = \frac{p}{p_0}$ **(4)** where p is pressure at the flying altitude, $\mathcal{h}$.

For the constant thrust motor (which is often encountered) the value of $\mathcal{T}_0$ is given by [6] $\mathcal{T}_0 = Q_e V_{ef} = \frac{G_e}{g} V_{ef} = \frac{\omega_0}{\tau_2} I_{sp}$ **(5)** where Q_e and G_e are the mass flow rate and the weight flow rate of the exhaust gasses in the exit of the engine's nozzle, respectively, V_{ef} is the "effective" speed of the gasses in the nozzle's exit, I_{sp} - the specific impulse of the engine, ω_0 - the total amount of fuel, τ_2 - the "ballistic" duration of the "active period" (of engine running), while $Q_e = \frac{G_e}{g}$, $G_e = \frac{\omega_0}{\tau_2}$, $I_{sp} = \frac{V_{ef}}{g}$ **(6)**

1.1.2 THE AERODYNAMIC DRAG

Within the hypotheses mentioned above, the aerodynamic drag, $\overline{R}$, is oriented along the velocity vector $\overline{V}$, but it acts in opposite direction, $\overline{R} = -R \cdot \frac{\overline{V}}{V} = -R \cdot \frac{V_x \bar{i} + V_y \bar{j} + V_z \bar{k}}{V}$ **(7)**, and the absolute value is $R = \frac{1}{2} \cdot \rho \cdot V^2 \cdot S \cdot C_x$ **(8)**

The density, ρ, and the specific weight, γ, at the altitude of the flight, $\mathcal{h}$, $\rho = \rho_0 \cdot \mathcal{H}(\mathcal{h})$, $\gamma = g \cdot \rho$, **(9)** where ρ_0 is the density of the air at the ground-level, while $\mathcal{H}(\mathcal{h})$ is the function which indicates the relative variation of air's density (or specific weight) with respect to the altitude, $\mathcal{h}$ [7, 6].

The symbol S of formula (8) is the reference aria, usually the aria of the transverse section of the body, $S = \frac{\pi \cdot d^2}{4}$, **(10)** where d is the maximum diameter of the transverse section of the body (fuselage), and C_x - the dimension-less aerodynamic drag coefficient.

Within the adopted hypotheses, at a given altitude, $\mathcal{h}$, the C_x coefficient is function of the Mach number during the flight, $M = \frac{V}{a}$, **(11)** where the speed of sound, a, at the flight altitude, $\mathcal{h}$, is given by $a = \sqrt{T \mathcal{R} \mathcal{K}}$, **(12)** in which $\mathcal{K}$ is the ratio of the specific air's heats, $\mathcal{K} = \frac{C_p}{C_v} = 1.405$, $\mathcal{R} = 287.1 \cdot \frac{J}{kg \cdot K}$ is the air's constant, and $T = T(\mathcal{h})$ is the air temperature at the flying altitude, $\mathcal{h}$ [7].

Substituting the numerical values, formula (12) becomes $a = 20.08 \cdot \sqrt{T(\mathcal{h})}$ **(12')**.

Within the same hypothesis, the value of C_x is expressed as a function $C_x = C_x(M, \mathcal{h})$ **(13)**.

It should be noticed that C_{xact}, which is the C_x function which correspond to the active phase of the flight, is different than C_{xpas}, which is the corresponding function during the passive phase. This is due to the fact that, during the active phase, the part of the drag produced by the vortices which appear at the posterior part of the rocket (at the bottom) while flying with engine off disappears due to jet of gasses exiting the engine.

Consequently, the drag coefficient of the rocket can be expressed as $C_x(M,h)=\begin{cases} C_{xact}(M,h), t_1 < t < t_2 \\ C_{xpas}(M,h), t > t_2 \end{cases}$ **(14)** where t_1 and t_2 are the value of the time at the beginning and the end of the active phase, respectively, while $C_{xact}=C_{xpas}-C_{xpost}$, **(15)** in which C_{xpost} (also noted C_{xp}) is the posterior drag coefficient (bottom drag) which disappears during the active phase, when the engine is running.

The function $C_x = C_x(M,h)$, which is usually given numerically, is established using experimental data and/or calculations.

Calculation in the ballistic practice often employ "drag functions" like

$$F(V,a)=\frac{\pi}{8000}\cdot\gamma_{0n}\cdot V^2\cdot C_x^{et}\left(\frac{V}{a}\right) \quad \textbf{(16)}$$

or

$$G(V,a)=\frac{\pi}{8000}\cdot\gamma_{0n}\cdot V\cdot C_x^{et}\left(\frac{V}{a}\right),$$

$$K\left(\frac{V}{a}\right)=\frac{\pi}{8000}\cdot\gamma_{0n}\cdot C_x^{et}\left(\frac{V}{a}\right) \quad \textbf{(17)},$$

where γ_{0n} is the specific weight of the air at ground level in "normal conditions" (for the ballistic standard atmosphere, according to [6], $\gamma_{0n}=1.206\frac{kgf}{m^3}$), and $C_x^{et}\left(\frac{V}{a}\right)\equiv C_x^{et}(M)$ is the "standard drag coefficient" or the "drag law" which corresponds to certain class of aerodynamic shapes [5, 6, 8, 11, 13, 15].

In the case, the C_x coefficient is expressed as $C_x(M)=i\cdot C_x^{et}(M)$, **(18)** where i is the shape index (or coefficient) of the rocket and it corresponds to the drag law which was chosen (usually using experimental data).

From formulae (16) and (17), we can obtain

$$F(V,a)=V\cdot G(V,a)=V^2\cdot K\left(\frac{V}{a}\right) \quad \textbf{(19)}$$

and

$$C_x^{et}(M)=\frac{8000}{\pi\cdot\gamma_{0n}\cdot V^2}F(V,a)=\frac{8000}{\pi\cdot\gamma_{0n}\cdot V}G(V,a)=\frac{8000}{\pi\cdot\gamma_{0n}}K\left(\frac{V}{a}\right) \quad \textbf{(20)}$$

where function $K\left(\frac{V}{a}\right)\equiv K(M)$ is a dimensional drag coefficient.

Consequently, the drag, defined by formula (18), can be expressed as

$$R=\frac{1}{2}\cdot\rho\cdot V^2\cdot S\cdot i\cdot\frac{8000}{\pi\cdot\gamma_{0n}\cdot V^2}F(V,a)=\frac{i\cdot S}{\pi\cdot g}\cdot\frac{\gamma}{\gamma_{0n}}\cdot 4000F(V,a) \quad \textbf{(21)}$$

where $\gamma=\rho\cdot g$ is the specific weight of the air at flight altitude.

If $H(h)=\frac{\gamma}{\gamma_{0n}}=\frac{\rho}{\rho_{0n}}$ **(22)** is the function that indicates the variation of the relative specific weight of the air with respect to the height, then the expression of R, expression (21), becomes

$$R=\frac{4\cdot i\cdot S}{\pi\cdot g}\cdot 10^3\cdot H(h)\cdot F(V,a)=\frac{4\cdot i\cdot S}{\pi\cdot g}\cdot 10^3\cdot H(h)\cdot V\cdot G(V,a)=\frac{4\cdot i\cdot S}{\pi\cdot g}\cdot 10^3\cdot H(h)\cdot V^2\cdot K\left(\frac{V}{a}\right) \quad \textbf{(23)}$$

Taking as reference surface the area of the maximum cross-section of rocket's body, formula (10), the expression of the drag, formula (23), becomes

$$R = \frac{i \cdot d^2}{\pi \cdot g} \cdot 10^3 \cdot H(h) \cdot F(V, a) =$$
$$= \frac{i \cdot d^2}{\pi \cdot g} \cdot 10^3 \cdot H(h) \cdot V \cdot G(V, a) = \quad \textbf{(24)}$$
$$= \frac{i \cdot d^2}{\pi \cdot g} \cdot 10^3 \cdot H(h) \cdot V^2 \cdot K\left(\frac{V}{a}\right)$$

The function $F(V, a)$, $G(V, a)$ and $K\left(\frac{V}{a}\right)$ are given numerically or in closed form (for various laws) [5,6,8,10,4,1], while the shape index, i, must correspond to the drag law which was adopted.

If the function of rocket's drag coefficient is available, then, in equation (23) and (24), the shape index must be chosen $i = 1$, while $K\left(\frac{V}{a}\right)$ is replaced by

$$K\left(\frac{V}{a}, h\right) = \frac{\pi}{8000} \cdot \gamma_{0n} \cdot C_x(M, h) \quad \textbf{(25)}$$

1.1.3 THE WEIGHT

As presented in [9], it is considered that the weight, $\overline{G}$, is oriented toward the center of the Earth (point P, ***figure 1***), so

$$\overline{G} = m \cdot \overline{g} = m \cdot \left(g_x \cdot \overline{i} + g_y \cdot \overline{j} + g_k \cdot \overline{k}\right), \quad \textbf{(26)}$$

where m is the mass of the rocket; the weight's acceleration vector, $\overline{g}$, at the current location O of the rocket in flight [9], is

$$\overline{g} = -g \cdot \frac{\overline{PO}}{\left|\overline{PO}\right|} = -g \cdot \frac{x \cdot \overline{i} + \left(y + R_p\right) \cdot \overline{j} + z \cdot \overline{k}}{\sqrt{x^2 + \left(y + R_p\right)^2 + z^2}} \quad \textbf{(27)}$$

If the distance $\left|\overline{PO}\right|$ between the rocket and the Earth's center is written as $\left|\overline{PO}\right| \equiv \mathcal{D}_{PO} \equiv \mathcal{D}_{CP} \equiv r_p = \sqrt{x^2 + \left(R_p + y\right)^2 + z^2}$, **(28)** then the weight's acceleration vector becomes

$$\overline{g} = -g \cdot \left(\frac{x}{\mathcal{D}_{CP}} \cdot \overline{i} + \frac{R_p + y}{\mathcal{D}_{CP}} \cdot \overline{j} + \frac{z}{\mathcal{D}_{CP}} \cdot \overline{k}\right). \quad \textbf{(29)}$$

In point O, located at the distance r_p and at the latitude λ from the center of the Earth, the value of g can be obtained from Expression (14) [9] (taking into account Equation (7) as well).

Neglecting the small terms and combining Equations (14) and (7), we have

$$g \cong a_p - \Omega_p^2 \cdot r_p \cdot \cos^2\lambda = a_p \cdot \left(1 - \frac{\Omega_p^2 \cdot r_p}{a_p} \cdot \cos^2\lambda\right) =$$
$$= \frac{f \cdot M}{r_p^2} \cdot \left(1 - \frac{\Omega_p^2 \cdot r_p^3}{f \cdot M} \cdot \cos^2\lambda\right) \quad \textbf{(30)}$$

Now, $r_p = R_p + h$, where, usually, $R_p = R_p(\lambda, \mu)$ is the distance between the center of the Earth and the surface of the Earth, along the geocentric vertical which passes through O, while h is the altitude of point O, so Formula (30) becomes

$$g = \frac{f \cdot M}{\left(R_p + h\right)^2} \cdot \left[1 - \frac{\Omega_p^2 \cdot \left(R_p + h\right)^3}{f \cdot M} \cdot \cos^2\lambda\right] \quad \textbf{(31)}$$

For altitudes h between zero and up to the order of magnitude of tenths of kilometers, and even higher, Equation (31) becomes [6]

$$g \cong \frac{f \cdot M}{\left(R_p + h\right)^2} \cdot \left[1 - \left(\frac{\cos\lambda}{17}\right)^2\right] \quad \textbf{(32)}$$

At the ground level, where $h = 0$, from Equation (31) we have

$$g_P = \frac{f \cdot M}{R_P^2} \cdot \left[1 - \frac{\Omega_P^2 \cdot R_P^3}{f \cdot M} \cdot \cos^2\lambda\right] \quad \textbf{(33)}$$

so

$$\frac{g}{g_P} = \left(\frac{R_P}{R_P + h}\right)^2 \left[\frac{1 - \frac{\Omega_P^2 \cdot \left(R_P + h\right)^3}{f \cdot M} \cdot \cos^2\lambda}{1 - \frac{\Omega_P^2 \cdot R_P^3}{f \cdot M} \cdot \cos^3\lambda}\right] \quad \textbf{(34)}$$

The fraction which appears in the brackets in the above equation is practically equal to 1 for flying altitudes h of up to several hundred kilometers, so in such cases Formula (34) is

$$g = g_P \cdot \frac{R_P^2}{\left(R_p + h\right)^2} = g_P \left(1 + \frac{h}{R_p}\right)^{-2}, \quad \textbf{(35)}$$

which allows for the calculation of the variation of g with respect to the height.

In such situations, the series expansion of the brackets in Equation (35) (using the binomial formula) is

$$g = g_P \cdot \left(1 - 2 \cdot \frac{h}{R_P} + 3 \cdot \frac{h^2}{R_P^2} - 4 \cdot \frac{h^3}{R_P^3} + \ldots\right) \quad \textbf{(36)}$$

For heights of the order of magnitude of tenths of kilometers (up to 100 km) it can be admitted that

$$g \cong g_P \cdot \left(1 - 2 \cdot \frac{h}{R_P}\right). \quad \textbf{(37)}$$

For points located on the Earth surface or close to it (at altitudes of the orders of tenths of kilometers, up to approximately 100 km) calculations can also employ the formula used in geodesics [14]

$$g = 9.806059 - 0.025028 \cdot \cos 2\lambda - 0{,}00000307h \quad \textbf{(38)}$$

which, for the latitude $\lambda = 45^0$ and $h = 0$, provides $g_P = 9.806059 \cdot m \cdot s^{-2}$. The standard value $g_P = 9.80665 \cdot m \cdot s^{-2}$ may be used instead, as well.

Correspondingly, g_P may also be estimated using [3]

$$g_P = 9.780573 \cdot (1 + 0.0052837 \cdot \sin^2 \lambda - 0.0000059 \cdot \sin^2 2\lambda) \quad \textbf{(39)}$$

which, for $\lambda = 45^0$, yields $g_P = 9.806354 \cdot m \cdot s^{-2}$.

1.2 THE INERTIAL CORIOLIS FORCE

The Coriolis force acting upon the rocket which has the velocity $\overline{V}$ with respect to the Earth, which, in turn, rotates with the angular transport velocity $\overline{\Omega}_P$ about the $P \cdot x_G$ [9] has the expression $\overline{F}_C = -m \cdot \overline{a}_C$, **(40)** where m is the mass of the rocket and $\overline{a}_C$ is the Coriolis acceleration, $\overline{a}_C = 2 \cdot \overline{\Omega}_P \times \overline{V}$, **(41)**
so

$$\overline{F}_C = -m \cdot 2 \cdot \overline{\Omega}_P \times \overline{V} = 2 \cdot m \cdot \overline{V} \times \overline{\Omega}_P. \quad \textbf{(42)}$$

The velocity $\overline{V}$ is expressed using the components in the Earth-linked frame O_1xyz, with respect to which the orbit is expressed as well.

Let $\left[\overline{\Omega}_P\right]_G$ be the column matrix containing the components of the $\overline{\Omega}_P$ vector on the axes of the geocentric Earth linked $Px_Gy_Gz_G$ frame (Figure 1),

$$\left[\overline{\Omega}_P\right]_G = \begin{bmatrix} \Omega_{Px_G} \\ \Omega_{Py_G} \\ \Omega_{Pz_G} \end{bmatrix} = \begin{bmatrix} \Omega_P \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(43)}$$

Since the axes of frame $O_1x_{G1}y_{G1}z_{G1}$ parallel to the axes of the frame $Px_Gy_Gz_G$ (***figure 1***), the $\left[\overline{\Omega}_P\right]_{G1}$ column matrix of the components of the vector $\overline{\Omega}_P$ with respect to the former frame is identical to $\left[\overline{\Omega}_P\right]_G$.

If $\left[\overline{\Omega}_P\right]$ is the column matrix containing the components of the vector $\overline{\Omega}_P$ with respect to the frame O_1xyz (***figure 1***), then

$$\left[\overline{\Omega}_P\right] = \begin{bmatrix} \Omega_{Px} \\ \Omega_{Py} \\ \Omega_{Pz} \end{bmatrix} = \Gamma_\beta \cdot \Gamma_{\lambda_1} \cdot \left[\overline{\Omega}_P\right]_{G1} = \Gamma_\beta \cdot \Gamma_{\lambda_1} \cdot \left[\overline{\Omega}_P\right]_G = \Gamma_{\lambda_1\beta} \cdot \left[\overline{\Omega}_P\right]_G \quad \textbf{(44)}$$

Taking into account Equation (22) [9], Expression (44) becomes

$$\begin{bmatrix} \Omega_{Px} \\ \Omega_{Py} \\ \Omega_{Pz} \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\lambda_1 & -\cos\beta\cos\lambda_1 & \sin\beta \\ \sin\lambda_1 & \cos\lambda_1 & 0 \\ -\sin\beta\sin\lambda_1 & \sin\beta\sin\lambda_1 & \cos\beta \end{bmatrix} \quad \textbf{(45)}$$

which gives the components:

$$\begin{aligned} \Omega_{Px} &= \Omega_P \cdot \cos\beta \cdot \cos\lambda_1 \\ \Omega_{Py} &= \Omega_P \cdot \sin\lambda_1 \\ \Omega_{Pz} &= -\Omega_P \cdot \sin\beta \cdot \cos\lambda_1 \end{aligned} \quad \textbf{(46)}$$

The expression of the Corollas force, (42), can be further expended as

$$\bar{F}_C = 2m \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ V_x & V_y & V_z \\ \Omega_{Px} & \Omega_{Py} & \Omega_{Pz} \end{vmatrix} = 2m[(V_y \cdot \Omega_{Pz} - V_z \cdot \Omega_{Py}) \cdot \bar{i}] + (V_z \cdot \Omega_{Px} - V_x \cdot \Omega_{Pz}) \cdot \bar{j} + (V_x \cdot \Omega_{Py} - V_y \cdot \Omega_{Px}) \cdot \bar{k} \quad \textbf{(47)}$$

Next, taking into account (46), the Coriolis force can be expressed as

$$\begin{aligned} \bar{F}_C &= 2m\Omega_P \cdot (-V_y \cdot \sin\beta \cdot \cos\lambda_1 - V_z \cdot \sin\lambda_1) \cdot \bar{i} + \\ &+ 2m\Omega_P \cdot (V_z \cdot \cos\beta \cdot \cos\lambda_1 + V_x \cdot \sin\beta \cdot \cos\lambda_1) \cdot \bar{j} + \\ &+ 2m\Omega_P \cdot (V_x \cdot \sin\lambda_1 - V_y \cdot \cos\beta \cdot \cos\lambda_1) \cdot \bar{k} \end{aligned} \quad \textbf{(48)}$$

which, defining the symbol $\bar{a}_{Fc} = -\bar{a}_C$ can be further written as

$$\bar{F}_C = m \cdot \bar{a}_{Fc} = m \cdot (a_{Fcx} \cdot \bar{i} + a_{Fcy} \cdot \bar{j} + a_{Fcz} \cdot \bar{k}), \quad \textbf{(49)}$$

where

$$\begin{aligned} a_{Fcx} &= 2 \cdot (V_y \cdot \Omega_{Pz} - V_z \cdot \Omega_{Py}) = \\ &= 2 \cdot \Omega_P \cdot (-V_y \sin\beta \cos\lambda_1 - V_z \sin\lambda_1) \\ a_{Fcy} &= 2 \cdot (V_z \cdot \Omega_{Px} - V_x \cdot \Omega_{Pz}) = \\ &= 2 \cdot \Omega_P \cdot (V_z \cos\beta \cos\lambda_1 + V_x \sin\beta \cos\lambda_1) \\ a_{Fcz} &= 2 \cdot (V_x \cdot \Omega_{Py} - V_y \cdot \Omega_{Px}) = \\ &= 2 \cdot \Omega_P \cdot (V_x \sin\lambda_1 - V_y \cos\beta \cos\lambda_1) \end{aligned} \quad \textbf{(50)}$$

2. THE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE ROCKET'S TRAJECTORY

The differential equations system of rocket's trajectory flying within a resistant medium and taking into account Earth's rotation and curvature is obtained from vector Equation (19) [9] and using Formulae (1), (2), (7), (26) and (49). Collecting the results, the trajectory with respect to the Earth-linked O_1xyz frame is described by:

$$\begin{cases} \dfrac{dV_x}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_x}{V} - \dfrac{R}{m} \cdot \dfrac{V_x}{V} + g_x + a_{Fcx} \\ \dfrac{dV_y}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_y}{V} - \dfrac{R}{m} \cdot \dfrac{V_y}{V} + g_y + a_{Fcy} \\ \dfrac{dV_z}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_z}{V} - \dfrac{R}{m} \cdot \dfrac{V_z}{V} + g_z + a_{Fcz} \\ \dfrac{dx}{dt} = V_x \\ \dfrac{dy}{dt} = V_y \\ \dfrac{dz}{dt} = V_z \end{cases} \quad \textbf{(51)}$$

where

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}\,. \quad \textbf{(52)}$$

Substituting the drag R with the general expression, Equation (8), and taking into account Equations (27) and (50), the system of differential equations of the rocket's trajectory becomes:

$$\begin{cases} \dfrac{dV_x}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_x}{V} - \dfrac{\rho S C_x}{2m} \cdot V \cdot V_x - g \cdot \dfrac{x}{\mathcal{D}_{cP}} + 2(V_y\Omega_{Pz} - V_z\Omega_{Py}) \\ \dfrac{dV_y}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_y}{V} - \dfrac{\rho S C_x}{2m} \cdot V \cdot V_y - g \cdot \dfrac{R_P + y}{\mathcal{D}_{cP}} + 2(V_z\Omega_{Px} - V_x\Omega_{Pz}) \\ \dfrac{dV_z}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_z}{V} - \dfrac{\rho S C_x}{2m} \cdot V \cdot V_z - g \cdot \dfrac{z}{\mathcal{D}_{cP}} + 2(V_x\Omega_{Py} - V_y\Omega_{Px}) \\ \dfrac{dx}{dt} = V_x \\ \dfrac{dy}{dt} = V_y \\ \dfrac{dz}{dt} = V_z \end{cases} \quad \textbf{(53)}$$

where the traction, $\mathcal{T}$, is determined as shown in **1.1**, the acceleration of the weight, g, can be obtained using Equation (14) [9] or (30), (31) or (36), the distance to the center of the Earth, $\mathcal{D}_{CP}$ is given by Equation (28), while Ω_{Px}, Ω_{Py}, and Ω_{Pz}, are calculated using Equation (46).

Expressing the drag from one of Equations (24), which are usually utilized in ballistics, and taking into account Equation (46), trajectory's differential equations system becomes

$$\begin{cases} \dfrac{dV_x}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_x}{V} - \dfrac{id^2}{mg} \cdot 10^3 H(h) K\left(\dfrac{V}{a}\right) \cdot V \cdot V_x - \\ \quad - g \cdot \dfrac{x}{\mathcal{D}_{cP}} - 2\Omega_P (V_y \sin\beta \cos\lambda_1 + V_z \sin\lambda_1) \\ \dfrac{dV_y}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_y}{V} - \dfrac{id^2}{mg} \cdot 10^3 H(h) K\left(\dfrac{V}{a}\right) \cdot V \cdot V_y - \\ \quad - g \cdot \dfrac{R_P + y}{\mathcal{D}_{cP}} + 2\Omega_P (V_x \sin\lambda_1 + V_x \sin\beta \cos\lambda_1) \\ \dfrac{dV_z}{dt} = \dfrac{\mathcal{T}}{m} \cdot \dfrac{V_z}{V} - \dfrac{id^2}{mg} \cdot 10^3 H(h) K\left(\dfrac{V}{a}\right) \cdot V \cdot V_z - \\ \quad - g \cdot \dfrac{z}{\mathcal{D}_{cP}} + 2\Omega_P (V_x \sin\lambda_1 - V_y \cos\beta \cos\lambda_1) \\ \dfrac{dx}{dt} = V_x \\ \dfrac{dy}{dt} = V_y \\ \dfrac{dz}{dt} = V_z \end{cases} \quad \textbf{(54)}$$

The latitude λ and the longitude μ of the rocket at a certain instant during the flight are obtained using Formulae (36) or (37) and (41) [9], while taking into account Expressions (34), (39) and (40) [9].

The altitude h, of the rocket during the flight is

$$h = \mathcal{I}_{cP} - R_P \quad \textbf{(55)}$$

where $\mathcal{I}_{cP}$ is the distance between the rocket and the center of the Earth [9], while R_P which is generally a function $R_P(\lambda,\mu)$, is the distance from the center of the Earth, P, up to a point P_P, located on the surface of the Earth, at the same latitude λ and longitude μ as the rocket.

Air pressure, p, and air density, ρ (or the specific weight, γ), at the altitude h (which occur in form (53) of the equations system) are obtained from Equations (4) and (9). The functions $\mathcal{B}(h)$ and $\mathcal{H}(h)$, which represent the relative variation with altitude of the pressure and density, respectively, can be expressed as

$$\mathcal{B}(h) = \frac{p}{p_0} = \frac{p_{0n}}{p_0} \cdot \frac{p}{p_{0n}} = \frac{p_{on}}{p_0} \cdot B(h)$$

$$\mathcal{H}(h) = \frac{\rho}{\rho_0} = \frac{\rho_{0n}}{\rho_0} \cdot \frac{\rho}{\rho_{0n}} = \frac{\rho_{0n}}{\rho_0} \cdot H(h) \quad \textbf{(56)}$$

where p_0 and ρ_0 are the ground-level air pressure and density p_{0n}, and ρ_{0n}, are the ground-level air pressure and density in normal conditions, while the functions $B(h) = \dfrac{\rho}{\rho_{0n}}$ and $H(h) = \dfrac{\rho}{\rho_{0n}} = \dfrac{\gamma}{\gamma_{0n}}$ are the functions that describe the relative variations of the air pressure and density (or specific weight), with respect to the altitude within the standard atmosphere model which was adopted [7, 12, 21].

For flight distances (ranges) of 40-50 km or even larger it can be admitted that R_P is a constant equal to the spherical homogeneous Earth ($R_P = 6371110 \cdot m$), without affecting meaningfully the accuracy of the results. In such a case the variation of g with respect to the latitude λ can also be neglected, however, its variation with respect to the altitude h must be taken into account.

The system of differential equations that was obtained represents the mathematical model of rocket's trajectory (in a resistant medium), which takes into account the daily Earth's rotation and Earth's curvature. The calculations are further performed via the numerical integration of this system.

The numerical integration of the differential equations system begins at the moment $t = t_1$, when the rocket looses contact with the launching device (e.g., the ramp), and the elements of the motions at the end of the motion of the launching device are used as the initial conditions, i.e.,

$$t = t_1, V_x = (V_x)_1 = V_1 \cos\theta_1,$$
$$V_y = (V_y)_1 = V_1 \cos\theta_1,\quad V_z = 0,\quad x = 0,\quad y = 0,$$
$$x = 0 \quad \textbf{(57)}$$

where V_1 and θ_1 are the speed while leaving the launcher and the launching angle, respectively.

During the passive phase of the flight, in other words after the end running time of the rocket engine, for $t > t_2$ (where t_2 is the extent of the active phase), the traction term in the equations of motion have to be voided ($\overline{\mathcal{T}} = 0$). The elements of the motion (state variables as well as other parameters) at the moment $t = t_2$ have to be recorded and the integration further proceeds using these ($t = t_2$) data as initial conditions.

The slope θ of the tangent to the trajectory with respect to the horizontal plane at the launching position, xO_1z is given by the expression

$$tg\theta = \frac{V_y}{\sqrt{V_x^2 + V_z^2}} \quad \textbf{(58)}$$

The equations of motion are usually integrated until a predetermined altitude, h, is reached on the descending side of the trajectory (when $\theta < 0$).

Calculation of the range requires integrating until $h = 0$ on the descending part of the trajectory.

Adequate choice of the initial conditions and of the end conditions allows the ordinary differential equations system obtained herein to be used for calculation of the active-reactive projectile's trajectory as well. So, if t_0 is the moment when the projectile exits the barrel, t_1 - the start of the rocket engine and t_2 corresponds to the end of the engine run, the ordinary differential equations system (with the appropriately selected boundary conditions) can be integrated choosing $\mathcal{T} = 0$ for the time-frame $[t_0, t_1]$, next setting, $\mathcal{T} \neq 0$, for the time-frame $[t_1, t_2]$ and, finally $\mathcal{T} = 0$, again for $t > t_2$, when the engine is no longer running.

The mathematical model presented in this paper can also be used (with the adequate adjustments) for calculating the trajectory of an aerospace vehicle within the dense atmosphere.

III. CONCLUSION

The mathematical model obtained is rendered in forms utilized in ballistics. This system of equations can also be used for the calculation of the trajectory of different rockets, active-reactive projectile, and also, with some adjustments, for the calculation of the trajectories of an aerospace vehicle.

For distances of 20-25 km, the inertial Coriolis force introduced by the Earth's rotation can be neglected. For bigger distances, the Coriolis force must be also considered. The variation of g with respect to the altitude h must be taken into account.

For flight distances (ranges) of 40-50 km or even larger, it can be admitted that R_P is a constant equal to the spherical homogeneous Earth ($R_P = 6371110 \cdot m$), without affecting, meaningfully, the accuracy of the results.

For distances over 50-60 km, the variation of g with respect to the latitude λ and altitude h must be taken into account.

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