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ANALYTICS SOLUTIONS FOR TRICOMI PROBLEM REGARDING MIXED ELLIPTIC-HYPERBOLIC EQUATIONS. APPLICATIONS IN TRANSONIC AERODYNAMICS

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Abstract: *In this paper is solved bounded Tricomi problem regarding mixed elliptic-hyperbolic equation positioned in x-o-y plan. For y>0 or y<0 canonical cases, is considered D1 domain where the equation is an elliptic type, respectively D2 domain where the equation is a hyperbolic type. By analytical completion from D1 to D2, for wave equation, there is determined exact solution for Dirichlet problem in D1, verifying problems' conditions. By conform representations in superior (upper) semi plan, other, more general, canonical D1 domains could be chosen (like semi circle, band etc.). Problems' applications are important in transonic aerodynamic where elliptic and hyperbolic equations correspond subsonic, supersonic or aero elasticity movement regimes.*

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MSC2010: *35M10, 76H20.*

1. INTRODUCTION

Let us consider the Lavrentiev-Bitzadze equation [1], [7]:

$$\frac{\partial^2 U}{\partial x^2} + \operatorname{sgn} y \frac{\partial^2 U}{\partial y^2} = 0. \qquad (1)$$

The equation is of elliptic type in the upper half-plane $D^+(y>0)$ and of hyperbolic type in the inferior half-plane $D^-(y<0)$ so we have:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0; \quad \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 0. \qquad (2)$$

Let consider $A(-1,0)$ and $B(1,0)$ the intersections of the *Ox* axis with the characteristic curves that are passing through point $C(0,-1)$. We have:

$$(AC): x + y + 1 = 0; \ (BC): x - y - 1 = 0 \qquad (3)$$

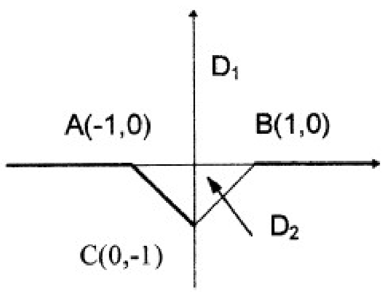

Figure 1.

We note $D_1 \equiv D^+$ and $D_2 \equiv D^-$ the interior of ABC triangle.

2. MAIN RESULTS

The Tricomi mixed boundary value problems for equation (1) require to find a function $U = U(x, y)$ of class $C^2(D)$, bounded at infinity in D_1 domain, continue on S and satisfying [5], [10];

$$U(x,0) = h_1(x),\ x \in (-\infty,-1);$$
$$U(x,0) = h_2(x).x \in (1,+\infty). \qquad (4)$$
$$U|_{(AC)} = P(x, y) = P(x,-x-1) = P(x) = P^*(y) \qquad (5)$$

The functions $h_i(x)$ and $P(x)$ are integrable. We are going to find the solution $U(x, y) = U^+$ in D_1 and $U(x, y) = U^-$ in D_2. On (AB) from the continuity we have $U^+(x) = U^-(x)$. In D_2 domain the Lavrentiev-Bitzadze equation become the wave equation and the solution can be write [2], [5]:

$$U^-(x, y) = \Phi(x + y) + \Psi(x - y) \qquad (6)$$

where Φ and Ψ are arbitrary derivable functions. From the condition (5) on the characteristic (AC) we have:

$$P(x) = \Phi(-1) + \Psi(2x + 1) \qquad (7)$$

and because Ψ is an arbitrary function we get:

$$\Psi(x) = P\left(\frac{x-1}{2}\right) - \Phi(-1). \qquad (8)$$

From (6) we can see the form of the solution:

$$U(x, y) = \Phi(x + y) + P\left(\frac{x-y-1}{2}\right) - \Phi(-1) \quad (9)$$

The continuity on (AB) impose:

$$U(x, y = 0) = U(x,0) = = \Phi(x) + P\left(\frac{x-1}{2}\right) - \Phi(-1) \qquad (10)$$

Here the Symmetry Principle of Schwarz was applied knowing that the solution $F(z)$ can be prolonged analytically on the inferior half-plane.

In the D_2 domain using (9) we can write:

$$\frac{\partial U}{\partial y} = \Phi'(x + y) - \frac{1}{2}P'\left(\frac{x-y-1}{2}\right) \qquad (11)$$

In order to find Φ let us determine the form of solution in D_1 domain where $U(x, y)$ is harmonic. In this case considering the harmonic conjugate $V(x, y)$ with $V(-1) = 0$ and the holomorphic function $F(z) = U(x, y) + iV(x, y)$ the Cauchy-Riemann relations hold true:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\phi}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \qquad (12)$$

We make use of equations (11) and (12) together with Symmeny Principle applied to $F(z)$. From continuity we have on (AB):

$$\left.\frac{\partial V}{\partial x}\right|_{y-0} = -\left.\frac{\partial U}{\partial y}\right|_{y=0} = -\Phi'(x) + \frac{1}{2}P'\left(\frac{x-1}{2}\right) \quad (13)$$

and performing one integration,

$$V(x,0) = -\Phi(x) + P\left(\frac{x-1}{2}\right) + \Phi(-1). \qquad (14)$$

Substitution of equation (14) in (10) yields on (AB):

$$U(x,0) + V(x,0) = 2P\left(\frac{x-1}{2}\right).$$

Thus in D_1 domain we must find a holomorphic function $F(z) = U + iV$ knowing on $x'Ox$:

$$U(x,0) = h_1(x),\ x \in (-\infty,-1);$$
$$U(x,0) = h_2(x),\ x \in (1,+\infty).$$

The last two equations can be rewritten in the form

$$aU(x) + bV(x) = g(x)$$

where

$$a = 1,\ b = 0,\ g(x) = h_{1,2}(x),$$
$$x \in (-\infty,-1) \cup (1,+\infty);$$
$$a = 1,\ b = 1,\ g(x) = 2P\left(\frac{x-1}{2}\right),\ x \in [-1,1].$$

The problem to be solved in order to find $F(z)$ is of Hilbert type. If $U^-(x,0)$ is known then from the continuity condition $U^+(x, y=0) = U^*(x) = U^-(x,0)$ we obtain:

$$\Phi(x) = U^*(x) - P\left(\frac{x-1}{2}\right) + \Phi(-1)\ . \qquad (15)$$

Relation (15) is the general form of $\Phi(x)$ and substituting in (9) we find in D_2 the solution:

$$U^-(x,y) = U^*(x+y) + P\left(\frac{x-y-1}{2}\right) - P\left(\frac{x+y-1}{2}\right) \qquad (16)$$

In order to find $F(z)$ we split the Hilbert problem in two problems. Let be

$$F(z) = f(z) + f^*(z),$$

where

$$f(z) = u + iv,\ f^*(z) = u^* + iv^*,$$
$$U = u + u^*, V = v + v^*.$$

Problem 1. Find in D_1 domain the holomorphic function $f(z)$ knowing on x' Ox axis:

$$U(x,0) = h_i(x), x \in (-\infty,-1) \cup (1,+\infty)\ ;$$
$$U(x,0) = 0,\ x \in [-1,1].$$

Problem 2. Find in D_1 domain the holomorphic function $f(z)$ knowing:

$$u^*(x,0) = 0, x \in (-\infty,-1) \cup (1,+\infty);$$
$$u^*(x) + v^*(x) = g(x),\ x \in [-1,1].$$

The solution for the Problem 1 is given by Cisotti formula:

$$f^*(z) = u + iv = \frac{1}{\pi i}\int_{-\infty}^{-1}\frac{h_1(t)}{t-z}dt + \frac{1}{\pi i}\int_{1}^{+\infty}\frac{h_2(t)}{t-z}dt + ik,\ k \in \mathbf{R}. \qquad (17)$$

If on segment $(-1,1)$ the substitution $t = \frac{1}{s}$ is considered then the evaluation of the integrals is much easier. The Hilbert Problem 2 is bring to a Dirichlet problem. Dividing the condition $au^* + bv^* = g(x)$ by $\sqrt{a^2 + b^2}$, we obtain:

$$u^* \cos \mu x + v^* \sin \mu x = g(x);\ \mu = \frac{1}{4}.$$

According to [4] along with the irrational function

$$H(z) = (z+1)^{1-\mu}(z-1)^{\mu},$$

where analytic in D_1, we consider:

$$S(z) = R(z) + iI(z) = \frac{f^*(z)}{H(z)}\ . \qquad (18)$$

Let us determine in D_1 the analytic function $S(z)$ satisfying the boundary conditions:

$$Re\{S(z)\} = 0,\ x \in (-\infty,-1) \cup (1,+\infty);$$
$$Re\{S(z)\} = \frac{g(x)\sqrt{2}}{(1+x)^{\frac{3}{4}}(1-x)^{\frac{1}{4}}},\ x \in (-1,1).$$

Using Cisotti formula and substituting t in (18), we get:

$$f^*(z) = u^*(x,y) + iv(x,y) =$$
$$= \frac{2\sqrt{2}(z+1)}{\pi i}\left(\frac{z-1}{z+1}\right)^{\frac{1}{4}}\int_{-1}^{1}\frac{P\left(\frac{t-1}{2}\right)}{(1+t)^{\frac{3}{4}}(1-t)^{\frac{1}{4}}}\frac{dt}{t-z} + ik$$

where k is a real constant. (19)

Thus with u(x, y) the real part of (17), (19) the solution is $U^{+}(x,y)=u(x,y)+u^{*}(x,y)$ in D_1 and for $y=0$ we obtain $U(x,y=0)=U^{*}(x)$ resulting $U^{*}(x)$ which is used in (16).

If the $h_i(x), i=1,2$ and P(x) are rational functions, the integrals in (18) and (19) can be easily calculated using the residue theorem.

3. CONCLUSIONS

Thus we have found the two solutions $U^{+}(x,y)$ and $U^{-}(x,y)$ for D_1 and D_2 domains. These solutions satisfied the boundary conditions (4) and (5).

The general case when the frontier of D_1 domain is a Liapunov curve passing through A to B and the segment (AB) is readily amenable to the case solved by meaning of a conformal mapping of domain D_1 onto one of the following canonical domains: upper half-plane, semicircle, strip.

If D_1 domain is bounded by a semicircle $\{x^2+y^2=1, y>0\}$ and the segment AB, then the mapping of D_1 domain onto upper half-plane is:

$$Z=\frac{2z}{1+z^2}$$

If D_1 domain is bounded by an infinite quadrilateral MABN, where MA and NB are parallel with Oy, $x\in(-1,1), y\in(0,+\infty)$, the conform mapping of D_1 onto upper half-plane is:

$$Z=\sin\frac{\pi}{2}z\,.$$

Immediate applications are known in transonic aerodynamics [3], [4], [6], [8], [10] where in D_1 the flow is subsonic, in D_2 is supersonic and (AB) is the sonic line. Also immediate applications are known in the study of magneto dynamics.

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