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# AN APPLICATION TO Mastroianni OPERATORS

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***Abstract.*** *A new linear positive operator is defined and studied resorting to the method of Mastroianni [2 ], [3].*

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## 1. Introduction

Motiveted by the Mastroianni′ s operators [2 ],[ 3] which were studied by O. Agratini, B. Della Vechia [1], we consider the sequence $\left(\Phi_{n,c}\right)_{n\in N}$, $c>0$ of real valued functions defined on [0, +∞) as

$$\Phi_{n,c}\left(x\right)=\left(\frac{c}{1+c}\right)^{ncx}, \quad c>0,\ x\geq 0,\ n\in N$$

.

These functions satisfy the following conditions, for each $n\in N$ and $c>0$:

(i). $\Phi_{n,c}\left(0\right)=1$,

(ii). $\Phi_{n,c}\in C^{\infty}\left[0,\infty\right)$,

(iii). $\Phi_{n,c}\left(x\right)$ is completely monotone, so that $\left(-1\right)^{k}\ \Phi^{(k)}{}_{n,c}\left(x\right)\geq 0,\ x\geq 0$,

(iv). $\Phi^{(k)}{}_{n,c}\left(x\right)=\left(nc\ln\frac{c}{1+c}\right)^{k}\Phi_{n,c}\left(x\right)$,

(v). $\lim\limits_{x\to\infty}x^{r}\Phi^{(k)}{}_{n,c}\left(x\right)=0,\ r\in N_{0}$.

Let

$$E_{2}\left[0,\infty\right):=\left\{ f\in C\left[0,\infty\right)\ \middle|\ (\exists)\ \lim_{x\to\infty}\frac{\left|f\left(x\right)\right|}{1+x^{2}}<\infty\ \right\}$$

be the space endowed with the norm $\left\|f\right\|_{*}=\sup\left\{\frac{\left|f\left(x\right)\right|}{1+x^{2}},\ x\geq 0\right\}$ which is a Banach space.

We define the operator $L_{n,c}:E_{2}\left[0,\infty\right)\to C\left[0,\infty\right)$ as

$$L_{n,c}\left(f;x\right)=\sum_{k=0}^{\infty}p_{n,k,c}\left(x\right)f\left(\frac{k}{n}\right) \text{ with} \tag{1.1}$$

$$p_{n,k,c}\left(x\right)=\frac{\left(-1\right)^{k}x^{k}\Phi_{n,c}^{\left(k\right)}\left(x\right)}{k!}=\frac{\left(-1\right)^{k}x^{k}\left(nc\ln\frac{c}{1+c}\right)^{k}\Phi_{n,c}\left(x\right)}{k!}. \tag{1.2}$$

Because

$$\sum_{k=0}^{\infty}p_{n,k,c}(x)=\Phi_{n,c}\left(x\right)\sum_{k=0}^{\infty}\frac{\left(-ncx\ln\frac{c}{1+c}\right)^{k}}{k!}=\Phi_{n,c}\left(x\right)\exp\left(-ncx\ln\frac{c}{1+c}\right)=1,$$

these operators preserve constant functions. Our aim is to find some approximation properties.

## 2. Main results

**Lemma 1.** *The moments of the operators $L_{n,c}\left(f;x\right)$ for $e_r\left(x\right)=x^r, r=0,1,2$ , $x\geq 0$ are given as*

$$L_{n,c}\left(e_0;x\right)=1,$$

$$L_{n,c}\left(e_1;x\right)=-\frac{\Phi_{n,c}'\left(0\right)}{n}e_1\left(x\right)=-cx\ln\frac{c}{1+c}, \tag{2.1}$$

$$L_{n,c}\left(e_2;x\right)=\frac{\Phi_{n,c}''\left(0\right)}{n^2}e_2\left(x\right)-\frac{\Phi_{n,c}'\left(0\right)}{n^2}e_1\left(x\right)=x^2c^2\ln^2\frac{c}{1+c}-\frac{x}{n}c\ln\frac{c}{1+c}.$$

*Proof.* By a simple computation it can easily be verified the identities of the lemma.

**Theorem 2.1.** *If $c=c\left(n\right)\to\infty,\ n\to\infty$ then $\lim_{n\to\infty}L_{n,\ c(n)}f=f$ uniformly on compact subsets of $\left[0,\infty\right)$ for any $\left(\forall\right)f\in E_2\left[0,\infty\right)$ .*

Indeed, taking $c=c\left(n\right)\to\infty,\ n\to\infty$ according to the well-known Bohmann-Korovkin theorem, relations of the lemma guarantee that $\lim_{n\to\infty}L_{n,\ c(n)}f=f$ uniformly on compact subsets of $\left[0,\infty\right)$ .

**Remark**. If $c\to\infty$ the operators (1.1) converge to classical Szasz-Mirakjan operators

$$S_n\left(f;x\right)=\sum_{k=0}^{\infty}e^{-nx}\frac{\left(nx\right)^k}{k!}f\left(\frac{k}{n}\right),x\geq 0.$$

Applying a classical result due to Shisha O. and Mond B. [4] we obtain the pointwise estimate

$$\left|L_{n,c}(f;x)-f(x)\right|\leq\left(1+\delta^{-1}\sqrt{L_{n,c}\left((e_1-xe_0)^2;x\right)}\right)\omega(f,\delta),x\geq 0$$

with

$$L_{n,c}\left((e_1-xe_0)^2;x\right)=x^2(1+2\frac{\Phi_n'(0)}{n}+\frac{\Phi_n''(0)}{n^2})-\frac{\Phi_n'(0)}{n^2}x=$$

$$=x^2(1+2c\ln\frac{c}{1+c}+c^2\ln^2\frac{c}{1+c})-x\frac{c}{n}\ln\frac{c}{1+c}.$$

These operators admitte a generalization of Durrmeyer type, because

$$\int_0^\infty p_{n,k}(x)dx=\int_0^\infty\frac{\left(-nxc\ln\frac{c}{1+c}\right)^k}{k!}\left(\frac{c}{1+c}\right)^{ncx}dx=\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\int_0^\infty x^k\left(\frac{c}{1+c}\right)^{ncx}dx=$$

$$=\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\cdot\frac{(-1)^k k!}{\left(nc\ln\frac{c}{1+c}\right)^k}\int_0^\infty\left(\frac{c}{1+c}\right)^{ncx}dx=-\frac{1}{nc\ln\frac{c}{1+c}}.$$

The Durrmeyer type operator asociated with the operator (1.1)-(1.2) become:

$$DL_{n,c}(f;x)=-\left(nc\ln\frac{c}{1+c}\right)\sum_{k=0}^\infty p_{n,k,c}(x)\int_0^\infty p_{n,k,c}(t)f(t)\,dt.$$

For these operators we have

$$DL_{n,\ c}\left(e_0;x\right)=1,$$

$$DL_{n,\ c}\left(e_1;x\right)=\left(-nc\ln\frac{c}{1+c}\right)\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\int_0^{\infty}\frac{\left(-nct\ln\frac{c}{1+c}\right)^k\Phi_{n,c}\left(t\right)}{k!}\ t\ dt=$$

$$=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\int_0^{\infty}t^{k+1}\Phi_{n,c}\left(t\right)dt=$$

$$=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\cdot\frac{\left(k+1\right)!}{\left(-nc\ln\frac{c}{1+c}\right)^{k+2}}=$$

$$=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{k+1}{\left(-nc\ln\frac{c}{1+c}\right)^2}=\frac{1}{-nc\ln\frac{c}{1+c}}\sum_{k=0}^{\infty}\left(k+1\right)p_{n,k,c}\left(x\right)=$$

$$=-\frac{1}{nc\ln\frac{c}{1+c}}nL_{n,c}\left(e_1;x\right)-\frac{1}{nc\ln\frac{c}{1+c}}=x-\frac{1}{nc\ln\frac{c}{1+c}}.$$

$$DL_{n,\ c}\left(e_2;x\right)=\left(-nc\ln\frac{c}{1+c}\right)\sum_{k=0}^{\infty}p_{n,k}\left(x\right)\int_0^{\infty}\frac{\left(-nct\ln\frac{c}{1+c}\right)^k\Phi_{n,c}\left(t\right)}{k!}\ t^2dt=$$

$$=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\int_0^{\infty}t^{k+2}\Phi_{n,c}\left(t\right)dt=$$

$$=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{\left(-nc\ln\frac{c}{1+c}\right)^k}{k!}\cdot\frac{\left(k+2\right)!}{\left(-nc\ln\frac{c}{1+c}\right)^{k+3}}=\left(-nc\ln\frac{c}{1+c}\right)\ \sum_{k=0}^{\infty}p_{n,k}\left(x\right)\frac{\left(k+2\right)\left(k+1\right)}{\left(-nc\ln\frac{c}{1+c}\right)^3}=$$

$$=\frac{1}{\left(-nc\ln\frac{c}{1+c}\right)^2}\ \sum_{k=0}^{\infty}\left(k+1\right)\left(k+2\right)p_{n,k,c}\left(x\right)=\frac{1}{\left(nc\ln\frac{c}{1+c}\right)^2}\left\{n^2L_{n,c}\left(e_2;x\right)+3nL_{n,c}\left(e_1;x\right)+2\right\}=$$

$$=x^2-\frac{4x}{nc\ln\frac{c}{1+c}}+\frac{2}{\left(nc\ln\frac{c}{1+c}\right)^2}.$$

If $c=c\left(n\right)\to\infty,\ n\to\infty$ then

$$\lim_{n\to\infty}DL_{n,\ c(n)}\left(f\right)=f,\ \left(\forall\right)f\in E_2\left[0,\infty\right)$$

uniformly on compact subsets of $\left[0,\infty\right)$.

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