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SOME FEATURES ABOUT STATIONARY DISTRIBUTION OF PROBABILITIES AND APPLICATIONS

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Abstract: *An important aspect of the distribution is that of the stationary. Please note that here we consider only a special class of Markov chains, and indeed, the term Markov chain should always be described by adding the clause of the constant probabilities of transition. We present an application as a result of the study about the Markov chains and the stationary distribution of the probabilities.*

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1. INTRODUCTION

The natural significance of stationary of a probability distribution becomes apparent if we imagine a large number of processes that occur simultaneously. Let it be, for example, N particles that are running independently the same type of random motion. At time n the medium number of particles in the state E_k is $Na_k^{(n)}$. With a stationary distribution these mean values remain constant and let observe (if N is large enough for applying the law of large numbers) a macroscopic equilibrium state maintained by a large number of passes in opposite directions. In physics, many statistical equilibriums are of this kind, ie they are the exclusively simultaneous result observing of many independent particles. It is a typical case of a symmetric random motion (or diffusion); if more particles are observed then, after a sufficiently long time, about half of them will be to the right of the origin, and the rest to the left of them. However, it is known (citing law sinus) that most individual particles do not behave such that, spending a greater part of the time on the same side of the origin.

2. SOME FEATURES ABOUT STATES OF A MARKOV CHAIN

Definition 1. [1], [4] *The state E_i leads to state E_j and we note $E_i \to E_j$ if there exist a number $n > 0$ such that $p_{ij}^{(n)} > 0$. We say that the state E_i communicate with the state E_j and we note that with $E_i \leftrightarrow E_j$ if $E_i \to E_j$ and $E_j \to E_i$.*

Definition 2. [1], [2], [5]. *A set of states C is closed if no state outside of C can not be touched by any E_j of the states of C. The smallest closed set that contains the C is called the* ***closure*** *of C.*

Definition 3. [1], [2], [5] *A state* E_k *which forms a single closed set is called* ***absorbing state***.

A Markov chain is called ***irreducible*** if there is no other closer set than the set of all states.

Obviously, C is closed if and only if $p_{jk} = 0$ whenever j is on C and k is outside of C, in this case, from Chapman-Kolmogorov equations we can see that $p_{jk}^{(n)} = 0$ for each n. So, it follows:

Theorem 1. [3], [7], [9] *If in* P^n *matrix we cut all the lines and columns that correspond to the outside states of the set C, we will obtain stochastic matrix which continues to maintain the fundamental relations of Chapman-Kolmogorov.*

That means that we can define on C a Markov chain and this subchain can be studied independently of all other states.

Remark 1. The state E_k is absorbing if and only if $p_{kk} = 1$; in this case the matrix of the last theorem is reduced to only one element.

The closure of a state E_j is the set of all states that can be reached from it (inclusive E_j). This remark can be reformulated as follows:

Definition 4. [8] *A Markov chain is* ***irreducible*** *if and only if each state can be reached from every other state.*

Application 1. [8] In order to determine all closed sets is sufficient to know which p_{jk} tend to zero and which are positive. Therefore, we use an asterisk to indicate the positive elements and we will consider the matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Bet on the fifth line an asterisk appears in fifth place and, so $p_{55} = 1$; the state E_j is absorbant. The third and the eighth lines contain only a positive element each, and it is obvious that E_3 and E_8 form a closed set. The crossings from E_1 are possible in E_4 and E_9, and from there only in E_1 , E_4 , E_9 . Consequently, the three states E_1 , E_4 , E_9 form another closed set.

We order now the states as follows:
$E_1' = E_5, E_2' = E_3, E_3' = E_8, E_4' = E_1, E_5' = E_9,$
$E_6' = E_4, E_7' = E_2, E_8' = E_7, E_9' = E_6$.

Elements of the matrix P are arranged in this way and, then, P takes the form

$$P' = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \end{pmatrix}$$

In this form the closed sets (E_1'), (E_2', E_3') and (E_4', E_5', E_6') appear clearly. From E_7' it is possible a pass in each of the three closed sets and, therefore, the closure of E_7' is the set of states E_1', E_2', E_3', E_4', E_5', E_6', E_7'. From E_8' it is possibile a pass in E_7' and in E_9' and, so, in every closed set. The closures of E_8' and of E_9' consist of all nine states.

Suppressing all the lines and all the columns from the outside of a closed set we get three stochastic matrices

$$(*), \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ * & 0 & 0 \end{pmatrix}$$

and *P'* does not contain any other stochastic submatrix.

We consider a fixed state E_j and we suppose that, initially, the system is in state E_j. Whenever the system passes through the state E_j the process is repeated from the beginning as it has been the first time. It is clear, therefore, that a return to E_j is a recurring event. If the system starts from a different state E_i then, passing through E_j becomes a *recurring event* delayed. Therefore, Markov chains appear as a special case of recurrent events simultaneously.

Each state E_j is characterized by its *recursive time distribution* $\left\{f_j^{(n)}\right\}$. Here $f_j^{(n)}$ is the probability that the first return to E_j occur at time *n*. From $p_{jj}^{(n)}$, we can calculate the probability $f_j^{(n)}$ using obvious recurrent relations

$$f_j^{(1)} = p_{jj},\ f_j^{(2)} = p_{jj}^{(2)} - f_j^{(1)} p_{jj},\ \dots,$$
$$f_j^{(n)} = p_{jj}^{(n)} - f_j^{(1)} p_{jj}^{(n-1)} - f_j^{(2)} p_{jj}^{(n-2)} - \dots - f_j^{(n-1)} p_{jj} \tag{1}$$

Relationships (1) express the fact that the probability of a *first* return to the state E_j, at the moment *n*, is equal with the probability of a return at the time *n*, minus the probability that the first return to take place at a time *v = 1,2, ..., n - 1*, and is followed by a repeated returning at time *n*.

The sum

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)} \tag{2}$$

is the probability that, starting from the state E_j, the system to get back to the state E_j.

Theorem 3. [6] *In an irreducible Markov chain, all of the states belong to the same class: they are all transitory, all zero-persistent states, or all non-zero persistent states. In each case they have the same period. In addition, each state may be achieved from any other state.*

Corollary 1. [6] *In a finite Markov chain there is no zero state and it is impossible that all of the states to be transient.*

3. ERGODIC PROPERTIES OF THE IRREDUCIBLE CHAINS

Definition 5. [8] *A probability distribution* $\{v_k\}$ *is called stationary if*

$$v_j = \sum_i v_i p_{ij} \tag{3}$$

If the initial distribution a_k is going to be stationary, then the absolute probabilities $\left\{a_k^{(n)}\right\}$ are independent of the time *n*, ie $a_k^{(n)} = a_k$.

The following theorem is often described as *a tendency towards equilibrium*.

Theorem 4. [7] *An irreducible periodically Markov chain belongs to one of the two classes:*

i) All states are either transient or are all null state; in this case $p_{jk}^{(n)} \to 0$ *when* $n \to \infty$ *for each pair j, k and there is no stationary distribution.*

ii) All the states are ergodic, ie

$$\lim_{n\to\infty} p_{jk}^{(n)} = u_k > 0 \tag{4}$$

Where u_k *corresponds to the medium recursive time of* E_k. *In this case* $\{u_k\}$ *is a stationary distribution.*

A weaker formulation can highlight the implications of this theorem. Thus, if (4) takes place, then for an arbitrary initial distribution a_k

$$a_k^{(n)} = \sum_j a_j p_{jk}^{(n)} \to u_k \tag{5}$$

Therefore, if there is a stationary distribution it is necessarily unique and the distribution at the time *n* tends to her independently from the initial distribution. The only alternative to this situation is that $p_{jk}^{(n)} \to 0$.

Demonstration

By the theorem 3, the relation (4) is keeping any time as long as its states are ergodic. For proofing the affirmation (ii), the above, we point out, first of all, that

$$\sum u_k \leq 1 \qquad (6)$$

This follows directly from the fact that, for fixed j and n, the quantities $p_{jk}^{(n)} (k = 1,2,3,...)$ have the sum equal with unity, such that $u_1 + u_2 + ... + u_N \leq 1$ for each N. For $n = 1$ and $m \to \infty$ we have the left side tending to u_k, and the general term from the right side of the sum tending to $u_v p_{vk}$. Adding an arbitrary number, but finite of terms we observe that

$$u_k \geq \sum_v u_v p_{vk} \qquad (7)$$

Summing these inequalities for all k we obtain, in each part the finite quantity, $\sum u_k$

This shows that in (7) the inequality is not possible and, therefore,

$$u_k = \sum u_j p_{jk} \qquad (8)$$

If we put $v_k = u_k \left(\sum u_j\right)^{-1}$ we find that v_k is a stationary distribution, such that there exist at least one distribution like that.

Let $\{v_k\}$ a certain distribution satisfying equality (3). Multiplying (3) by $p_{jk}^{(n)}$, and summing after j, we deduce, by induction, that

$$v_r = \sum_v v_k p_{vr}^{(n)} \qquad (9)$$

If $n \to \infty$, we obtain

$$v_r = (v_1 + v_2 + ...)u_r = u_r \qquad (10)$$

which completes the proof of point (*ii*). If states are transient or zero state and $\{v_k\}$ is a stationary distribution then equations (9) remain valid and $p_{vr}^{(n)} \to 0$ which is obviously impossible.

As a consequence, a stationary distribution may exist only in the ergodic case and the theorem is proved.

4. APPLICATION

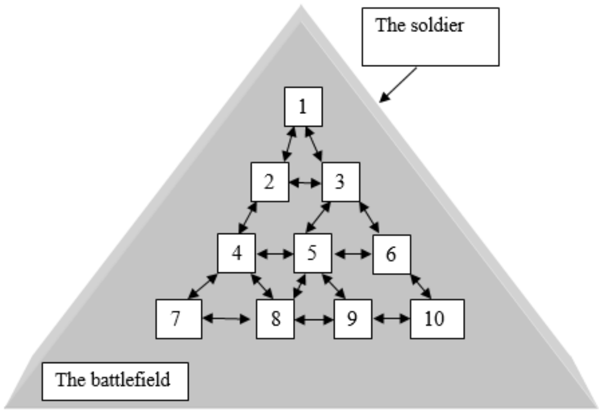

Fig. (1)

A soldier enters on the battlefield which contains 10 charging points of weapons. From each power point he can move to another point neighbor. He chooses with equal probabilities either supply points which are available. For example, from the no.1 building he moves with the same probability $\frac{1}{2}$, in the no. 2 and no. 3 buildings. From no.2 buildings he moves with the probability $\frac{1}{4}$ in buildings no.1, no.3, no.4 and no.5 etc.

We will determine the stationary distribution of probabilities (the limit probability) with each soldier is arming in every collecting point.

We note with E_n the collecting point in which one the soldier will arm at the time n. The chain $\{E_n\}$ is Markov, with the set of states 1, 2, … 10 and transition probabilities

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

All the states communicate with each other such that they form a single class (obviously positive). We have $p_i = y_i$; y_i obtained from the solving the system (8) with the conditions (10):

$$\begin{cases} y_1 + y_2 + y_3 + \ldots + y_{10} = 1 \\ y_1 = \frac{1}{4}y_2 + \frac{1}{4}y_3 \\ y_2 = \frac{1}{2}y_1 + \frac{1}{4}y_3 + \frac{1}{4}y_4 + \frac{1}{6}y_5 \\ y_3 = \frac{1}{2}y_1 + \frac{1}{4}y_2 + \frac{1}{6}y_5 + \frac{1}{4}y_6 \\ y_4 = \frac{1}{4}y_2 + \frac{1}{6}y_5 + \frac{1}{2}y_7 + \frac{1}{4}y_8 \\ y_5 = \frac{1}{4}y_2 + \frac{1}{4}y_3 + \frac{1}{4}y_4 + \frac{1}{4}y_6 + \frac{1}{4}y_8 + \frac{1}{4}y_9 \\ y_6 = \frac{1}{4}y_3 + \frac{1}{6}y_5 + \frac{1}{4}y_9 + \frac{1}{2}y_{10} \\ y_7 = \frac{1}{4}y_4 + \frac{1}{4}y_8 \\ y_8 = \frac{1}{4}y_4 + \frac{1}{6}y_5 + \frac{1}{2}y_7 + \frac{1}{4}y_9 \\ y_9 = \frac{1}{6}y_5 + \frac{1}{4}y_6 + \frac{1}{4}y_8 + \frac{1}{2}y_{10} \\ y_{10} = \frac{1}{4}y_6 + \frac{1}{4}y_9 \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1 = y_7 = y_{10} = \frac{1}{18} \\ y_2 = y_3 = y_4 = y_6 = y_8 = y_9 = \frac{1}{9} \\ y_5 = \frac{1}{6} \end{cases}$$

5. CONCLUSIONS

So if the soldier is in the initial point i with probability $p_i = y_i, 1 \le i \le 10$, then in every minute there exist the same probability $y_i \left(p_i^{(n)} = p_i = y_i\right)$ that the soldier to be in i point. On the other hand, even if the initial probabilities p_i are different of y_i the ergodic character of the chain assures us that after many moments, probability that the soldier to be the point j will be close to the limit probability, y_j. He will be found most frequently in the no.5 weapons collection point where he returns on average every 6 considered moments.

In conclusion we can say that it is usually easy in terms of comparison, to decide whether there is a stationary distribution and therefore if a given irreducible chain is ergodic.

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