USING THE ANALOGY IN TEACHING TETRAHEDRON GEOMETRY

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Abstract: *In the teaching tetrahedron geometry, we often meet similar properties to those of the triangle and that is why it is good to emphasize and use this analogy. The result will develop in students not only the functional thinking, but also the analog thinking.*

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MSC2010: *97D10, 51M04*

1. INTRODUCTION

While the triangle is determined by three non-collinear points, the tetrahedron is defined by four non-coplanar points. In the following we will make a brief, but very useful parallel between the two geometric notions.

2. PARALLEL BETWEEN TRINGLE AND TETRAHEDRON

We will follow Table 1, where we will find the properties of the triangle and tetrahedron:

Table 1.

No.	TRIANGLE	TETRAHEDRON
1.	$Area = \frac{B \cdot h}{2}$	$Volume = \frac{A_b \cdot h}{3}$
2.	The center of gravity in the triangle is distanced at 1/3 from the base and 2/3 from the peak.	The center of gravity in the tetrahedron is distanced at 1/4 from the base and ¾ from the peak.
3.	The center of the inscribed circle in the triangle is determined by the intersection of bisectors and $r = \frac{S}{p}$ where r is radius of the circle, $S$is the area and $p$is the semiperimeter of the triangle.	The center of the inscribed sphere in the tetrahedron is determined by the intersection of bisecting planes of the dihedral angles and $r = \frac{3V}{S}$ where r is radius of the sphere, $V$is the volume and $S$is the total area of the tetrahedron.
4.	The center of the circumscribed circle of the triangle is determined bythe intersection of perpendicular bisectors.	The center of the circumscribed sphere of the tetrahedron is determined by the intersection of the median planes of tetrahedron edges.

5.	In an equilateral triangle the three medians have equal lengths.	In a tetrahedron with equiareal faces the four medians have equal lengths.
6.	In an equilateral triangle the three altitudes have equal lengths.	In a tetrahedron with equiareal faces the four altitudes have equal lengths.
7.	Triangle angle bisector theorem: A bisector of an angle of a triangle divides the opposite side in two segments that are proportional to the other two sides of the triangle.	Dihedral angle bisector theorem in tetrahedron: A bisector half-plane of an dihedral angle in tetrahedron divides the opposite edge in two segments that are proportional to the other two surfaces of the dihedral angle.
8.	Cathetus theorem in the right triangle ABC with $m(\angle A)=90^\circ$ and $AD \perp BC$: $AB^2 = BC \cdot BD$ and similars.	An analogue of the Cathetustheorem in the tetrahedron $VABC$, with $VA \perp VB \perp VC \perp VA$ and $AH \perp (ABC)$: $A^2_{\Delta VAB} = A_{\Delta ABC} \cdot A_{\Delta HAB}$ andsimilars.
9.	The right triangle altitude theorem in the right triangle ABC with $m(\angle A)=90^\circ$ and $AD \perp BC$: $AD^2 = DC \cdot DB$ and similars.	An analogue of the right triangle altitude theorem in the tetrahedron $VABC$, with $VA \perp VB \perp VC \perp VA$ and $AH \perp (ABC)$: $A^2_{\Delta VCD} = A_{\Delta CAD} \cdot A_{\Delta HBC} = A_{\Delta CBD} \cdot A_{\Delta HAC}$ andsimilars.
10.	Pithagorean theorem in the right triangle ABC with $m(\angle A)=90^\circ$: $BC^2 = AB^2 + AC^2$	An analog of the Pithagorean theorem in an tetrahedron $VABC$, with $VA \perp VB \perp VC \perp VA$ and $AH \perp (ABC)$: $A^2_{\Delta ABC} = A^2_{\Delta VAB} + A^2_{\Delta VBC} + A^2_{\Delta VCA}$

In the following there will be proved some of the theorems presented in the previous table. The numbering will be kept and it will only refer to the theorems of the tetrahedron.

Now we will prove 7. from the Table 1. (A bisector half-plane of an dihedral angle in tetrahedron divides the opposite edge in two segments that are proportional to the other two surfaces of the dihedral angle).(**FIG. 1.**)

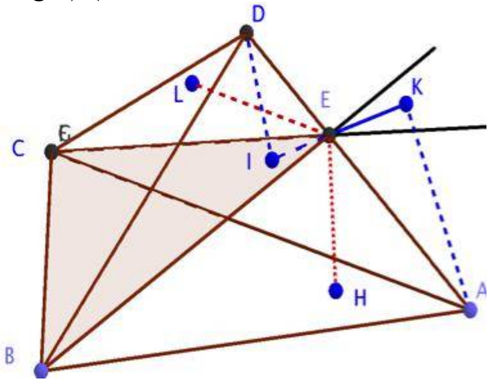

FIG. 1.

Let be E the point where the bisector half-plane intersects the opposite edge of the tetrahedron and $EL \perp (BCD), EH \perp (ABC)$. Because (BCE) is the bisector-half plane, the distances to the faces of the dihedral angle are the same, therefore $EL = EH$.

$$V_{DBCE} = \frac{S_{BCD} \cdot EL}{3}, V_{ABCE} = \frac{S_{ABC} \cdot EH}{3}$$

$$\frac{V_{DBCE}}{V_{ABCE}} = \frac{\frac{S_{BCD} \cdot EL}{3}}{\frac{S_{ABC} \cdot EH}{3}} = \frac{S_{BCD}}{S_{ABC}} \cdot \frac{EL}{EH} = \frac{S_{BCD}}{S_{ABC}} \tag{1}$$

Let $DI \perp (BCE), AK \perp (BCE)$. Therefore $DI \| AK$. We will prove that I, E, K are collinear. Let $\beta = (DI, AK)$ From $E \in AD \subset \beta$ and $E \in (BCE) \Rightarrow E \in \beta \cap (BCE)$.

But $I, K \in \beta \cap (BCE)$ and then $I, E, K \in \beta \cap (BCE)$, so they are collinear.

Volumes of the two tetrahedrons can also be expressed by:

$$V_{DBCE} = \frac{S_{CBE} \cdot DI}{3}, V_{ABCE} = \frac{S_{CBE} \cdot AK}{3}, \quad \frac{V_{DBCE}}{V_{ABCE}} = \frac{\frac{S_{CBE} \cdot DI}{3}}{\frac{S_{CBE} \cdot AK}{3}} = \frac{DI}{AK} \tag{2}$$

From $DI \| AK \Rightarrow \Delta DIE \approx \Delta AKE \Rightarrow \frac{DI}{AK} = \frac{IE}{KE} = \frac{DE}{AE}$ (3)

And from (1),(2)si (3)$\Rightarrow \frac{S_{BCD}}{S_{ABC}} = \frac{DE}{AE}$.

Now we will prove that in tetrahedron $VABC$ with $VA \perp VB \perp VC \perp VA$ the orthogonal projection of point V on the plane (ABC), H, is the orthocenter of the triangle ABC and the validity of the statements 8.9.10. from the Table 1:(**FIG. 2.)**

$A^2_{\Delta VAB} = A_{\Delta HAB} \cdot A_{\Delta ABC}$ (for 8.) (4)

$A^2_{\Delta VCD} = A_{\Delta CAD} \cdot A_{\Delta HBC}$ (for 9.) (5)

$A^2_{\Delta ABC} = A^2_{\Delta VAB} + A^2_{\Delta VBC} + A^2_{\Delta VCA}$ (for 10.) (6)

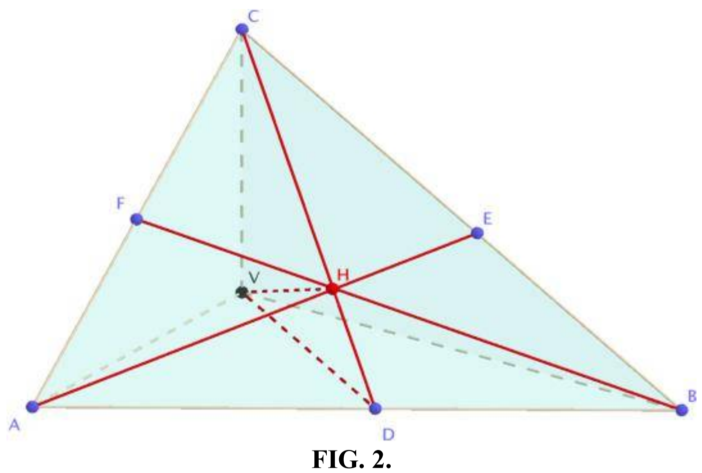

FIG. 2.

Let $VH \perp (ABC)$. We will prove H is the orthocenter of the triangle ABC. First we will prove that $CH \perp AB$. From $VC \perp VB$ and $VC \perp VA \Rightarrow VC \perp (VAB) \supset AB \Rightarrow VC \perp AB$.

From $VH \perp (ABC) \supset AB \Rightarrow VH \perp AB$

From $VC \perp AB$ and $VH \perp AB \Rightarrow AB \perp (VCH) \supset CH \Rightarrow AB \perp CH$

Similarity it can demonstrated that $BH \perp AC$ and then H is the orthocenter of the triangle ABC.

Now we will prove that the validity of the statement 8. fromthe Table 1 .

From $A^2_{\Delta VAB} = \left(\frac{AB \cdot VD}{2}\right)^2$ $A_{\Delta HAB} = \frac{AB \cdot HD}{2}$, $A_{\Delta ABC} = \frac{AB \cdot CD}{2}$

Using Cathetus Theorem in ΔVCD: $VD^2 = HD \cdot CD$ we have:

$$A^2_{\Delta VAB} = A_{\Delta HAB} \cdot A_{\Delta ABC} \Leftrightarrow \left(\frac{AB \cdot VD}{2}\right)^2 = \frac{AB \cdot HD}{2} \cdot \frac{AB \cdot CD}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{AB^2 \cdot VD^2}{4} = \frac{AB^2 \cdot (HD \cdot CD)}{4}$$

It will prove the validity of statement 9. From the Table 1:

From (4) Cathetus Theorem in the right triangle ΔVCD: $CV^2 = CD \cdot CH$ and $VD^2 = AD \cdot BD$ (The right triangle altitude theorem in the right triangle ΔVAB)

$$A^2_{\Delta VCD} = A_{\Delta CAD} \cdot A_{\Delta HBC} \Leftrightarrow \left(\frac{CV \cdot VD}{2}\right)^2 = \frac{CD \cdot AD}{2} \cdot \frac{CH \cdot BD}{2} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{CV \cdot VD}{2}\right)^2 = \frac{(CD \cdot CH) \cdot (AD \cdot BD)}{2} \Leftrightarrow \left(\frac{CV \cdot VD}{2}\right)^2 = \frac{CV^2 \cdot VD^2}{4}$$

Finally it will prove the validity of statement 10. from the Table 1:

From $A^2_{\Delta VAB} = A_{\Delta HAB} \cdot A_{\Delta ABC}$, $A^2_{\Delta VBC} = A_{\Delta HBC} \cdot A_{\Delta ABC}$, $A^2_{\Delta VAC} = A_{\Delta HAB} \cdot A_{\Delta ABC}$

Then $A^2_{\Delta VAB} + A^2_{\Delta VBC} + A^2_{\Delta VCA} = A_{\Delta HAB} \cdot A_{\Delta ABC} + A_{\Delta HBC} \cdot A_{\Delta ABC} + A_{\Delta HAC} \cdot A_{\Delta ABC} =$

$A_{\Delta ABC} \cdot \left(A_{\Delta HAB} + A_{\Delta HBC} + A_{\Delta HAC}\right) = A^2_{\Delta ABC}$.

CONCLUSIONS

The few analogies studied in this work open the way to the comparative study between plane geometry concepts and the concepts of the space geometry. Therefore, the knowledge of the plane geometry is not just an adjunct to the geometry of space, but also a rich source of inspiration and analogies.

REFERENCES

[1] ***GeoGebra 5. Available at http:/.geogebra.en.softonic.com/mac, accessed on March 23, 2016;
[2] D. Branzei, M. Miculita, *Tetrahedron triangle analogies,* Paralela 45, Pitesti 2000;
[3] L. Nicolescu, V. Boskoff, *Applied Geometry Problems*EdituraTehnica, Bucuresti 1990;
[3] H. V. Stoiculescu, I.M. Popa, I. Popa *Geometry and Trigonometry fotXth Grad,* Didactic and Pedagogic Publishing, Bucuresti, 1990