OPTIMAL CONTROL IN STABILIZING THE DYNAMICS OF SHIPS AND MULTI-PROPELLED MISSILES

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Abstract: *This paper deals with automatic stabilization of rockets, submarines or satellites dynamics. This stabilization is based on relay-type automatic regulators, by using the minimal time criterion for optimal control with the Pontreagiune extremal principle. In this study, the state variables are the rotation angles and the control function has 2 components, which are appearing because of lateral rolling perturbations. Finaly, numeric-analytical studies are approached, and the results are graphically presented.*

Key words: *optimal control, control function, extreme principle of Pontreaguine, absolute stability.*

MSC2010: *34H05, 49K35, 93C15, 93C73, 93D10.*

1. INTRODUCTION

The goal of an optimal control problem (O.C.P.) of a dynamic system is to determine a set of state variables, of certain control and driving functions satisfying an optimization criterion, performing the extremization of a quality index in this way. This performance index is a functional depending on these elements and time and spatial restrictions.. Practical applications requirements for these functionals are optimal controls of the following type: achievement of minimal time, minimum fuel consumption, energy, to achieve extreme performance [4,10,13]. Dynamical systems from different domains are generally represented generally by nonlinear equations with parameters, while internal or external disturbances occur leading to unstable solutions related to a free balance state. The stabilization of these regimes is done by using automatic controls that actually fast reacts for optimal control and routing [3, 5-7, 12, 14]. Lurie [8], [4], [13] and Popov [11], [4], [3] methods are known to automatically adjust the absolute stabilization, with applications. This paper, for optimal control of stabilizing angular velocities of aircraft and missiles, deals with the "Minimum Time Criteria" and the Pontreaguine extremal [9], [1], [7], with results and studies in different applications [5], [6], [7]. The optimal control function will have 2 components: $u(u_1, u_2)$.

2. OPTIMAL CONTROL IN ANGULAR SPEED STABILIZATION REGARDING FLUVIAL OR SPATIAL NAVES

We will consider a multi-propelled nave with axial-cylinder symmetry, as in figure 1. We choose a system of axes $(Ox_1x_2x_3)$ as principal axes of inertia, where O is the masses, as a solid body rigidly fixed in point O. The nave rotate with the angular velocity

$\overline{\omega}(\omega_1, \omega_2, \omega_3)$, where ω_i are the rotation speeds around the axes Ox_i, $i = \overline{1,3}$; Ox_3 is the axis of longitudinal symmetry.

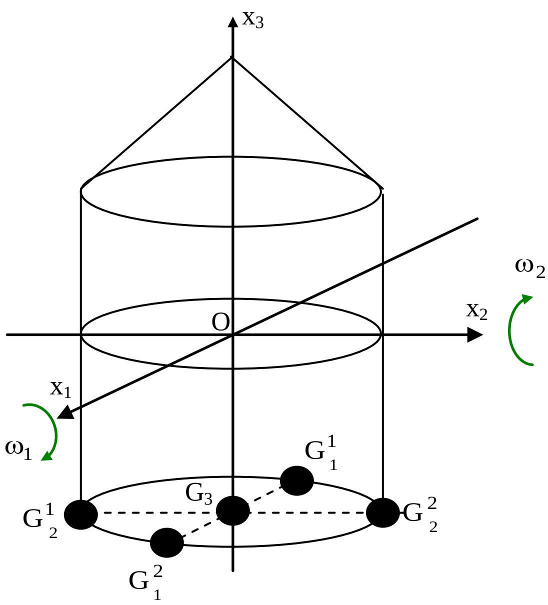

FIG. 1: A multi-propelled nave

Symmetrical with these axes the ship has turbojet propulsion generators $G_1(G_1^1, G_1^2)$ on Ox_1 with two nozzles, $G_2(G_2^1, G_2^2)$ on Ox_2 and generator G_3 on Ox_3, with traction purpose. These reactive nozzles can create moments, being accompanied by gas dynamics wings, integrating gyroscopes, small jet shutters that can help to guide and stabilize the angular velocities regime. The controller [3], [5-7] is equipped with sensors and microprocessors for data processing and it may (with a rapid response) control the disrupted regime for optimal stabilization [12], [14]. Disturbances considered here may be due to turbines fuel, meteo external agents or environmental density. These naves may be rockets, spacecraft, capsule, modules, mega-drones or submarines, torpedoes, etc [8], [12], [14].

The angular velocities are $\omega_1 = x_1(t)$, $\omega_2 = x_2(t)$, $\omega_3 = x_3(t)$, the inertia momentums of the body are I_1, I_2, I_3 - symmetrically $(I_1 = I_2 = I)$, see figure 1.

We write the equations of angular velocities disturbed by external moments $M_i(t)$ [10], [12], [14]:

$$\begin{cases} I_1\dot{x}_1 = (I_2 - I_3)x_2x_3 + M_1(t) \\ I_2\dot{x}_2 = (I_3 - I_1)x_1x_3 + M_2(t) \\ I_3\dot{x}_3 = (I_1 - I_2)x_1x_2 + M_3(t) \end{cases} \tag{1}$$

We suppose that moments $M_i(t)$ are caused by propelling forces $G(g_1, g_2, g_3)$:

$$h_1 = l \cdot g_1(t),\ h_2 = l \cdot g_2(t),\ h_3 = h \cdot g_3(t) \tag{2}$$

Taking into consideration the symmetry: $I_1 = I_2 := I$, we have:

$$\begin{cases} \dot{x}_1 = \dfrac{I - I_3}{I} x_2x_3 + \dfrac{l}{I} g_1(t) \\ \dot{x}_2 = -\dfrac{I - I_2}{I} x_3x_1 + \dfrac{l}{I}\ \ g_1(t) \\ \dot{x}_3 = \dfrac{h}{I_3} g_3(t) \end{cases} \tag{3}$$

System (1) with $M_i(t) = 0$ is in stable equilibrium around $O(0, 0, 0)$ (undisturbed).

Let's suppose that at $t_0 = 0$ we have the disturbed position

$$x_1(0) = \alpha_1,\ x_2(0) = \alpha_2,\ x_3(0) = \alpha_3 \tag{4}$$

If the traction $g_3(t)$ is known, than we have

$$x_3(t) = \alpha_3 + \int_0^t \frac{h}{I_3} g_3(\tau)d\tau \tag{5}$$

This shows that x_3 can be controlled independently of x_1 and x_2, but x_3 can influence in (3) variables x_1 and x_2. We suppose that the rapid reaction response time is short and x_3 may be considered constant $x_3 = \alpha_3$, so $g_3 \cong 0$. We also suppose that forces g_1, g_2 are bounded

$$|g_1(t)| \le L,\ |g_2(t)| \le L \tag{6}$$

In this case the system (3) is linearized and we introduce the control function $u(u_1(t), u_2(t))$ to control optimum stabilization of disturbed solution to O(x_1=0, x_2=0) for system (7) in minimal time.

$$\begin{cases} \dot{x}_1 = \omega x_2 + k u_1(t) \equiv f_1 \\ \dot{x}_2 = -\omega x_1 + k u_2(t) \equiv f_2 \end{cases} \tag{7}$$

where:

$$\begin{cases} \omega = \dfrac{I - I_3}{I}\alpha_3;\ \ \omega > 0 \\ k = \dfrac{l}{I}L; \qquad k > 0 \\ u_i(t) = \dfrac{g_i(t)}{L};\ \ i = 1, 2 \end{cases} \tag{7'}$$

We note that state equations (7) in the phase space $(x_1 O x_2)$, $X(x_1(t); x_2(t))$, with control parameters $(u_1(t); u_2(t))$satisfy the conditions:

$$\begin{cases} -1 \le u_1(t) \le 1 \\ -1 \le u_2(t) \le 1 \end{cases}; \qquad U := [-1,1] \tag{8}$$

and hence the allowable plan U ($u_1 O u_2$) is a compact square.

The technical sense in equations (7) for U_i (t) is to find forces $g_i(t) \leftrightarrow u_i(t)$ to reduce speeds $(x_1, x_2) \rightarrow (0, 0)$ on optimal paths starting from $M_0(\alpha_1, \alpha_2)$ at time t_0 to reach the final target O(0, 0) at the time $t_f > t_0$ so the transfer time to be minimal (o.c.p.) [2], [3], [9], [7].

3. MINIMAL TIME CRITERION. EXTREMUM PRINCIPLE

Let's consider a system described by the state equations

$$\dot{x}_i = f_i(t, x(t), u(t)),\ t \in [t_0, t_1] \subseteq \mathbf{R}_+,\ i = \overline{1, n} \tag{9}$$

where the state function is $x(t) = (x_1, x_2, ..., x_n), x_i \in X \subset \mathbf{R}^n$ and the control function $u(t) = (u_1, u_2, ..., u_m) \subset U \subset \mathbf{R}^m$ with $m \le n$.

Functions f_i meet the regularity conditions and U is the allowable domain of parameters u_i(t). System (9) respects the given initial conditions (I):

$$\text{(I)}\ \ x_i(t_0) = x_i^0 \in R, i = \overline{1, n} \tag{10}$$

determining the disturbed initial state $X^0 \in S^0$, where S^0 is the variety on which X^0 is fixed.

Assuming that Cauchy problem (9) (10) has $x_i = x_i(t, t_0, x^0, u), t \ge t_0$ as unique solution, we request that this trajectory transfer the system in the state $X^1 \in S^1$, where X^1 is fixed on S^1 (target (final) state - usually steady state in the final moment t_1 = t_f horizon pool) - $t_0 \le t_1$:

$$\text{(F)}\ \ x_i(t_1) = x_i^1 \in R, i = \overline{1, n} \tag{11}$$

The final moment t_1 will be determined using "minimal time criterion" - rapid response $\min\limits_{u\in U}(t_1 - t_0) = t^*$, extremizing the performance index J = J [u] [1], [2], [9].

Let's consider the index functional, see [1], [9], [10]:

$$J(u) = \int_{t_0}^{t_1} f_0(x, u, t)dt \tag{12}$$

where f_0 is a characteristic function (Lagrangean) with:

$$\dot{x}_0(t) \equiv f_0(t, x, u); x_0(t_0) = C_0 \tag{13}$$

The optimal control problem (P.C.O.) is to determine an optimal admissible command $u^* \in U$ to extremize (12) so that the original system (9) (10) (I) is transferred to the final system (11) (F) in minimal time (minimum criteria).

"Extreme Pontriaguine principle" (P.E.) will be calling to solve it. We auxiliary introduce the multipliers $\lambda(t) = (\lambda_0(t), \lambda_1(t), ..., \lambda_n(t))$ as non-null solution of the adjunct system [1] [9] [7] [10]:

$$\dot{\lambda}_i = -\sum_{j=0}^{n}\left(\frac{\partial f_j}{\partial x_i}\right)_0 \lambda_i \ ; \ \lambda_i(t_0) = c_i, i = \overline{1, n} \tag{14}$$

associated with (9)…(13) with arbitrary constants c_i (but not all of them arbitrary), which will finally become the controller parameters.

We note that (14), if it's linearized: $\dot{X} = AX + BU$ in the final null position (O) vicinity, $A = (a_{ij}) = \left(\frac{\partial f_i}{\partial x_j}\right)_0$ i.e. $\dot{\lambda} = -A'\lambda$, where $\dot{\lambda} = -A'\lambda$, A' is the transposed matrix.

We consider the lagrangean like f_0 and (12) :

$$L(t, x(t), u(t)) \equiv f_0(x, u, t) \equiv 1, \dot{x}_0 = f_0 \equiv 1 \tag{15}$$

$$J(u) = \int_{t_0}^{t_1} Ldt = t_1 - t_0 \, ; \min_u J(u) = J(u^*) = \min_u (t_1 - t_0) = t^* \tag{16}$$

We build the generalized Hamiltonian [9] [10] associated with (11) (12) (13) (14):

$$H(t, x(t), \lambda(t), u(t)) = \lambda_0 \dot{x}_0 + \sum_{i=1}^{n} \lambda_i \dot{x}_i \tag{17}$$

where $\dot{x}_0 = f_0 \equiv 1$, with $\dot{\lambda}_0 = \left(\frac{\partial f_0}{\partial x}\right)_0 \lambda \equiv 0, \lambda_0 \equiv C_0$

From (11) and (17) we have:

$$H \equiv \lambda_0 + \sum_{i=1}^{n} \lambda_i f_i(t, x, u) \tag{18}$$

and we built the canonic attached and adjunct system [1] [2] [7] [9]:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \tag{19}$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, i = \overline{1, n}, t \in [t_0, t_1] \tag{20}$$

with $u_i \in [-1, 1]$ and initial conditions $x_i(t_0) = x_i^0$, $\lambda_i(t_0) = c_i$. This system is:

$$\dot{x}_i = f_i(t, x, u); \dot{\lambda}_i = -\lambda_i \sum_{j=1}^{n}\left(\frac{\partial f_j}{\partial x_i}\right)_0 \tag{21}$$

with 2n unknowns: x_i, λ_i and 2n conditions, where u* is previously and optimal chosen. "Pontreaguine minimum principle" theorem ([1] [2] [9]): A necessary condition for the existence of an optimal solution $u^* \in U \quad (x_i \in [-1,1])$ minimizing (11), $\min\limits_u J(u) = J(u^*) \equiv \min\limits_u (t_1 - t_0) = t^*$ associated with (9), (10), (11), (18), where $\min\limits_u H(t, x, \lambda, u) = H^*(t, x, \lambda, u^*) = 0, \forall (t, x, \lambda)$ is that trajectories x_i, λ_i respect (19), (20), (21) $\forall t \in [t_0, t^*]$ with:

$$H(t, x, \lambda, u) \geq H^*(t, x, \lambda, u^*) \geq 0; \min(H) = H^*(t^*, x^*, \lambda^*, u^*, C) = 0 \tag{22}$$

Remarks:

1) We may take $\lambda_0 = c_0 \equiv 1$ and H(t) is minimized determining the vector $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ so that the speed $\dot{X} = (\dot{x}_i)$ projection on λ vector to be minimum: $\min\left(\sum\limits_{i=1}^{n} \lambda_i \dot{x}_i\right)$.

2) After building H, (17) and (18) with H = H(u), we find u* with $-1 \leq u_i \leq 1$, generally with $\frac{\partial H}{\partial u} = 0$; but, if H is linear in u, $H = a + bu_1 + cu_2$, then, according to linear programming with $H \geq 0$ in compact square $u(t) \in U$ included in (H, u1, u2) space, the minimum H(u*)=0 will be in the square tips $u^* = \{(u_1^* = -1, u_2^* = 1), (u_1^* = 1, u_2^* = -1)\}$; the solutions will be $x = x(t, u^*), \lambda = \lambda(t, u^*)$.

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4. ANGULAR SPEEDS STABILIZATION OPTIMAL CONTROL

We still apply the algorithm (9) - (22) to (1) – (8); and build Hamiltonian (17) (18) associated with the system (7):

$$H = 1 + \lambda_1 \dot{x}_1 + \lambda_2 \dot{x}_2 = 1 + \lambda_1 \omega x_2 - \lambda_2 \omega x_1 + \lambda_1 k u_1 + \lambda_2 k u_2 \geq 0 \tag{23}$$

We note that $H = H(u_1, u_2) = a + bu_1 + cu_2$ is linear and positive in the compact square $u_1(t) \in [-1;1]$, $u_2(t) \in [-1;1]$; so in the space (H, u_1 , u_2) H has a null minimum in the square tips: $H \geq H_{\substack{\min \\ k \in U}} = H(u_1^*, u_2^*) = 0$

$$\text{So, if } u_1^* = -\operatorname{sgn}(\lambda_1); u_2^* = -\operatorname{sgn}(\lambda_2) \tag{24}$$

the trajectories $C_1^{\pm}(u_1 = 1, u_2 = -1)$ or $C_1^{\mp}(u_1 = -1, u_2 = 1)$ will be obtained, and they will tend to origin $O(x_1 = 0, x_2 = 0)$.

The system is autonomic and u^* is pulse-type. It results that the controller will be a relay-type one, acting with or without commutation [1] [2] [7]. We solve the canonic system (13)(14), effectively (7) with initial conditions $x_i(t_0 = 0) = \alpha_i, i = 1, 2$ and $u \in U$:

$$\begin{cases} \omega x_1 - u_2^* k = (\omega \alpha_1 - u_2^* k) \cos \omega t + (\omega \alpha_2 + u_1^* k) \sin \omega t \\ \omega x_2 + u_1^* k = -(\omega \alpha_1 - u_2^* k) \sin \omega t + (\omega \alpha_2 + u_1^* k) \cos \omega t \end{cases} \tag{25}$$

$$\left(x_1 - \frac{u_2^* k}{\omega}\right)^2 + \left(x_2 + \frac{u_1^* k}{\omega}\right)^2 = \left(\alpha_1 - \frac{u_2^* k}{\omega}\right)^2 + \left(\alpha_2 + \frac{u_1^* k}{\omega}\right)^2 \tag{26}$$

We note that these trajectories are circles with centers $O^*\left(\frac{u_2^* k}{\omega}, -\frac{u_1^* k}{\omega}\right)$ and radius $R^2 = \left(\alpha_1 - \frac{u_2^* k}{\omega}\right)^2 + \left(\alpha_2 + \frac{u_1^* k}{\omega}\right)^2$, but they are not reaching the target point O(0,0) for $\forall(\alpha_1, \alpha_2)$. If in (26) $x_1 \equiv 0, x_2 \equiv 0$, we will get for $(\alpha) = (x)$ the compatible circles – a bi-local problem.

$$\left(\omega x_1 - u_2^* k\right)^2 + \left(\omega x_2 + u_1^* k\right)^2 = 2k^2; R = \frac{k\sqrt{2}}{\omega}; O_0^*\left(\frac{u_2^* k}{\omega}, -\frac{u_1^* k}{\omega}\right) \tag{27}$$

$$\begin{cases} x_1 - \frac{u_2^* k}{\omega} = \frac{k\sqrt{2}}{\omega}\cos\theta \\ x_2 + \frac{u_1^* k}{\omega} = \frac{k\sqrt{2}}{\omega}\sin\theta \end{cases}; \ \theta = \theta_0 + \omega(t - t_0) \tag{28}$$

We note that the origin is on the circles from this family and their centers are on the first bisecting line (Oz_1) of the system x_1Ox_2 or on the second one (Oz_2), with the (z_1Oz_2) axis system, see figure 2a. Choosing the optimal circles depends on the optimal control u* so that $H^*(u^*) \geq 0$; if the X system would be rotated with $\frac{\pi}{4}$: $z \to z = x \cdot e^{i\frac{\pi}{4}}$, we note that from $H(u^*) \geq 0$, with (23) and (24) it results the trajectory on the upper (towards Oz_1) half-circles $\{C_1^*\}$ from the first quadrant and the lower half-circles, $\{C_2^*\}$ from the third quadrant, i.e.: $\{C_{10}^{\mp} : u_1^* = -1, u_2^* = 1\} \cup \{C_{20}^{\pm} : u_1^* = 1, u_2^* = -1\}$, with centers respectively: $O_{10}^{\mp}\left(\frac{k}{\omega}, \frac{k}{\omega}\right)$ and $O_{20}^{\pm}\left(-\frac{k}{\omega}, -\frac{k}{\omega}\right)$. Choosing the initial point $M^0(\alpha_1^0, \alpha_2^0) \in C_{10}^{\mp}$ in the moment t_0, corresponds to the angle θ_0 towards Ox_1:

$$\tan\theta_0 = \frac{\alpha_2^0 - \frac{k}{\omega}}{\alpha_1^0 - \frac{k}{\omega}};$$

It may be observed on figure 2a that $\theta_0 \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, with $\theta = \omega t + \frac{\pi}{4}$.

$$\theta_0 \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right], t_0 \in \left[0, \frac{\pi}{\omega}\right], t^* = \frac{1}{\omega}\left(\frac{5\pi}{4} - \theta_0\right), \alpha_1^0 > 0, \alpha_2^0 > 0 \tag{29}$$

Analog and asymmetrically for the circle $C_2^{\pm}$.

Remark: The optimal trajectories are periodical $T = \frac{2\pi}{\omega}$; $\{C_{1j}^{\mp}\} \cup \{C_{2j}^{\pm}\}$ are tangent half-circles, with centers $O_{1j}^{\mp}\left((2j+1)\frac{k}{\omega}, (2j+1)\frac{k}{\omega}\right), O_{2j}^{\pm}\left(-(2j+1)\frac{k}{\omega}, -(2j+1)\frac{k}{\omega}\right), j = 0,1,2,...$ and radius $R = \frac{k\sqrt{2}}{\omega}$; for example, if the starting point is $M_{0j} \in C_{1j}^{\mp}$, then the minimal time will be $t^* = j\frac{k}{\omega} + \frac{1}{\omega}\left[\frac{5\pi}{4} - \theta_0(M_{0j})\right]$, without relay commutation.

There are situations when the starting point is not on the small half-circles, for example $P_0(\gamma_1, \gamma_2)$ in the third quadrant in figure 2b. In this case we choose from $C_{2j}^{\pm}$ a half-circle $O_{2j}^{\pm}$ with radius R_Γ denoted $\Gamma^{\pm}$. This circle crosses one of the half-circles $C_{1j}^{\mp}$. This is the commutation moment $(u_i \to -u_i, i = 1, 2)$, the new trajectory starting from Q and ending in origin.

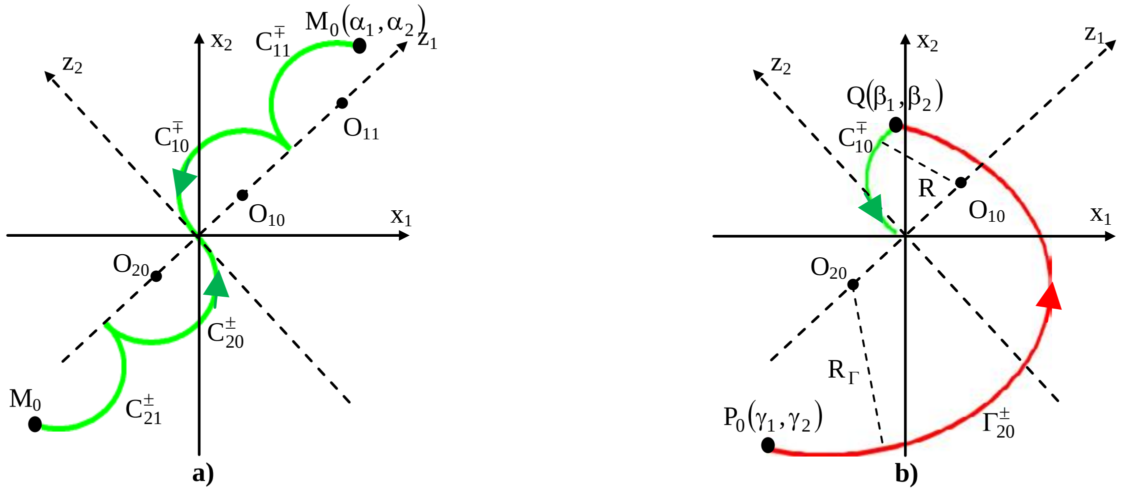

FIG. 2: Optimal trajectories
a) With no relay commutation b) with relay commutation

We solve the adjunct system (14-20) with H from (23):

$$\dot{\lambda}_1 = -\omega\lambda_2, \dot{\lambda}_2 = \omega\lambda_1, \lambda_1(t=0) = c_1^+, \lambda_2(t=0) = c_2^- \tag{30}$$

$$\lambda_1 = c_1 \cos\omega t + c_2 \sin\omega t; \lambda_2 = c_1 \sin\omega t - c_2 \cos\omega t \tag{31}$$

$$\lambda_1^2 + \lambda_2^2 = c_1^2 + c_2^2; \lambda_1 = \sqrt{c_1^2 + c_2^2}\cos(\omega t - \varphi_0); \lambda_2 = \sqrt{c_1^2 + c_2^2}\sin(\omega t - \varphi_0) \tag{32}$$

$$\tan\varphi_0 = \frac{c_2}{c_1}; \varphi_0 = a\tan\left(\frac{\lambda_2^0}{\lambda_1^0}\right) \tag{33}$$

We note that the adjunct system solution (31) describes a circle (32) and the period is the same as $C_{10}^{\mp}$, $C_{20}^{\pm}$ semicircles one, so $C_{10}^{\mp}$ semicircle description coincides with the description in the same direction of the semicircle upper $\lambda_2 > 0$ and $C_{20}^{\pm}$ description coincides with lower semicircle $\lambda_2 < 0$ in the plane $(\lambda_1 O \lambda_2)$.
Choosing the signs for u* and λ, c_1, c_2 was determined by interpreting the semi-circles in (z_1Oz_2) as follows:

$$\begin{array}{l} \{C_{10}^{\mp}; u_1^* = -1; \lambda_1 > 0; c_1 > 0; u_2^* = 1; \lambda_2 < 0; c_2 < 0 \text{ pentru } z_2 \geq 0\} \text{ şi} \\ \{C_{20}^{\pm}; u_1^* = 1; \lambda_1 < 0; c_1 < 0; u_2^* = -1; \lambda_2 > 0; c_2 > 0 \text{ pentru } z_2 \leq 0\} \end{array} \tag{34}$$

4.1. Optimal Control of the Un-commutated System [1], [2], [7]

Let's suppose that in the initial moment t_0 the system is in $M_0(\alpha_1, \alpha_2)$ on $C_{10}^{\mp}$ or $C_{20}^{\pm}$, and must reach the (target point) $O(0.0)$ in minimum time t_1^*; $\alpha_1 > 0$, $\alpha_2 > 0$ or

$\alpha_1 < 0$, $\alpha_2 < 0$, α_i given in the system (x_1Ox_2) on $C_{10}^{\mp}$ or $C_{20}^{\pm}$ (without relay switching).

$$C_{10}^{\mp}: \left(x_1 - \frac{k}{\omega}\right)^2 + \left(x_2 - \frac{k}{\omega}\right)^2 = \left(\frac{k\sqrt{2}}{\omega}\right)^2; u_1^* = -1; u_2^* = 1; t \in \left[0; \frac{\pi}{\omega}\right] \tag{35}$$

M_0 corresponds to the angle:

$$\theta_0 = \begin{cases} a\tan\dfrac{\alpha_2 - \frac{k}{\omega}}{\alpha_1 - \frac{k}{\omega}} & \dfrac{\pi}{4} \le a\tan\dfrac{\alpha_2 - \frac{k}{\omega}}{\alpha_1 - \frac{k}{\omega}} \le \dfrac{\pi}{2} \\ \pi + a\tan\dfrac{\alpha_2 - \frac{k}{\omega}}{\alpha_1 - \frac{k}{\omega}} & -\dfrac{\pi}{2} \le a\tan\dfrac{\alpha_2 - \frac{k}{\omega}}{\alpha_1 - \frac{k}{\omega}} \le -\dfrac{\pi}{4} \end{cases} ; \frac{\pi}{4} \le \theta_0 \le \frac{5\pi}{4} \tag{36}$$

We find $t_0 = \dfrac{\theta_0}{\omega}$ and as in $O(0.0)$ $t_f = \dfrac{\theta_f}{\omega} = \dfrac{5\pi}{4\omega}$, it results $t_{min}^* = \dfrac{1}{\omega}\left(\dfrac{5\pi}{4} - \theta_0\right)$ (37)

$$H_0^*\left(t_f, X^*, U^*, \lambda^*\right) = 1 + k\lambda_2 - k\lambda_1 = 0 \tag{38}$$

With (23), (31-33): $H_0^*(c_1, c_2) = 1 + k\sqrt{c_1^2 + c_2^2}\left(\sin\left(\omega t^* - \varphi_0\right) - \cos\left(\omega t^* - \varphi_0\right)\right) = 0$, by choosing characteristic values c_1, c_2 with (33) on microprocessors. Analog if the system starts with $M_0 \in C_{20}^{\pm}$.

4.2. Optimal Control of the Commutated System

Let's suppose that trajectories (26) are starting at t_0 from $P_0(\gamma_1, \gamma_2)$, $P_0 \in \Gamma_2^{\pm}$, not reaching in O; they intersect with the switching curves $C_1^{\mp}$ in Q and will optimally arrive in $O(0.0)$. P_0 is in the third quadrant, $\gamma_1 < 0$, $\gamma_2 < 0$, with $\theta_0(P_0) \in \left[\dfrac{5\pi}{4}, \dfrac{9\pi}{4}\right]$. We consider the circle $\Gamma_{20}^{\pm}$ (figure 2b), with center $O_{20}\left(-\dfrac{k}{\omega}, -\dfrac{k}{\omega}\right)$ and radius $R_\Gamma^2 = \left(\gamma_1 + \dfrac{k}{\omega}\right)^2 + \left(\gamma_2 + \dfrac{k}{\omega}\right)^2$; $\dfrac{k\sqrt{2}}{\omega} < R_\Gamma < \dfrac{3k\sqrt{2}}{\omega}$. It results the equation:

$$\Gamma_{20}^{\pm}: \left(x_1 + \frac{k}{\omega}\right)^2 + \left(x_2 + \frac{k}{\omega}\right)^2 = \left(\gamma_1 + \frac{k}{\omega}\right)^2 + \left(\gamma_2 + \frac{k}{\omega}\right)^2 \tag{39}$$

The Trajectories (39) $\Gamma_{20}^{\pm}$ will intersect (35) $C_{10}^{\mp}$ in $Q(\beta_1, \beta_2)$, i.e. the commutation point. We solve the system (39)(35): by subtracting the equations the radical axis (A) is obtained, so we'll solve the system {(35), (A)} [1][2][7].

$$(A)\, x_1 + x_2 = \frac{\omega}{4k}\left(\gamma_1^2 + \gamma_2^2 + + \frac{2k}{\omega}(\gamma_1 + \gamma_2)\right) := E > 0; x_1 > 0; x_2 > 0 \tag{40}$$

From (35) and (40) it results $x_1 = \beta_1$; $x_2 = \beta_2$; $Q(\beta_1, \beta_2)$

$$x_1 = \frac{1}{2}(E - \delta) > 0;\ x_2 = \frac{1}{2}(E + \delta) > 0, \text{ where } \delta = \sqrt{\frac{4k}{\omega}E - E^2} \tag{41}$$

Generally, $\theta = \omega t + \frac{5\pi}{4}$; we will find $t_0(P_0)$ and $\theta_0(P_0)$, then $t_1(Q)$ and $\theta_1(Q)$:

$$\theta_0(P_0) = \begin{cases} \pi + a\tan\dfrac{\gamma_2 + \frac{k}{\omega}}{\gamma_1 + \frac{k}{\omega}} & \dfrac{\pi}{4} \le a\tan\dfrac{\gamma_2 + \frac{k}{\omega}}{\gamma_1 + \frac{k}{\omega}} \le \dfrac{\pi}{2} \\ 2\pi + a\tan\dfrac{\gamma_2 + \frac{k}{\omega}}{\gamma_1 + \frac{k}{\omega}} & -\dfrac{\pi}{2} \le a\tan\dfrac{\gamma_2 + \frac{k}{\omega}}{\gamma_1 + \frac{k}{\omega}} \le -\dfrac{\pi}{4} \end{cases} ; \ \frac{5\pi}{4} \le \theta_0 \le \frac{9\pi}{4} \tag{42}$$

$$\theta_1(Q) = 2\pi + a\tan\frac{\beta_2 + \frac{k}{\omega}}{\beta_1 + \frac{k}{\omega}} \tag{42'}$$

$$t_1^* = t_1^1 - t_0^1 = \frac{1}{\omega}\left[\theta_1(Q) - \theta_0(P_0)\right] \tag{43}$$

which is the P_0Q arc travel time.

With $Q(\beta_1, \beta_2)$ determined, we pass on arc QO (on the circle $C_{10}^{\mp}$) respecting the determinations (35) – (38) and replacing (α_1, α_2) with (β_1, β_2) in the second stage.

$$\theta_0^2(Q) = \begin{cases} a\tan\dfrac{\beta_2 - \frac{k}{\omega}}{\beta_1 - \frac{k}{\omega}} & \dfrac{\pi}{4} \le a\tan\dfrac{\beta_2 - \frac{k}{\omega}}{\beta_1 - \frac{k}{\omega}} \le \dfrac{\pi}{2} \\ \pi + a\tan\dfrac{\beta_2 - \frac{k}{\omega}}{\beta_1 - \frac{k}{\omega}} & -\dfrac{\pi}{2} \le a\tan\dfrac{\beta_2 - \frac{k}{\omega}}{\beta_1 - \frac{k}{\omega}} \le -\dfrac{\pi}{4} \end{cases} ; \ \frac{\pi}{4} \le \theta_0^2 \le \frac{5\pi}{4} \tag{44}$$

$$t_0^2(Q) = \frac{\theta_0^2}{\omega}; \ t_2^*(Q) = \frac{1}{\omega}\left(\frac{5\pi}{4} - \theta_0^2\right) \tag{44'}$$

The final minimal time for the trajectories (35) and (39) is

$$t^* = t_1^* + t_2^* \tag{45}$$

and $H^*(O) \equiv 0$, $H^*\left(t_2^*, X^*, U^*, \lambda^*\right) = H(c_1, c_2) \equiv 0$ is used after (38).

Numerical and graphic applications were developed for the two above mentioned situations.

Remark:

If the speeds system is amortized - resistance terms (elastic damped oscillator) appear - trajectories are spiral arcs. The study of this case is analogous to those presented in the paper.

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