TRANSITION PROBABILITY MODELING FOR QUANTUM OPTICS

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Abstract: *The phenomenon of stimulated optical transition in anisotropic crystals that has been experimentally studied in our quantum lab; both its mathematical modeling and numerical simulation are approached. In order to evaluate the stimulated transition probability a revision of the perturbation theory equations is performed. For these equations the states spectrum of a quantum system being perturbated by an other system (in the frame of the quantum physics Hilbert space) is considered. Our formalization acts in accordance with the formal framework of information theory (characterized by: entropy, conditional entropy and mutual information) applied to two sources that interact, one being a perturbation of the other. In order to perform the numerical simulation the analytical relations are systematized. Some particular temporal patterns of the perturbation (e.g. (quasi) -rectangular or (envelope) -sinus, mono-pulse) and their corresponding transition probabilities are analyzed, then normalized and afterwards graphically represented using MathCAD.*

Keywords: *Quantum Optics, transition probability, MathCAD*

1. INTRODUCTION

The mathematical modeling and computer numerical simulation are necessary steps in the design of engineering quantum optics applications for quantum information processing.

But [1] "*these subjects follow either a semi-classical approach (often oversimplified), or a full quantum approach (often too difficult)*". This is the motivation for this revised physical modeling based on mathematical rigorous description.

In order to numerically evaluate the stimulated transition probability of a system when interacting with an other system (in the frame of the quantum physics Hilbert space [2]) a revision of the perturbation theory equations is performed in agreement with the requirements of our software tool (mathCAD).

This formalization is in accordance with the formal framework of the information theory applied to the interaction of two sources, one of each being the perturbation of the other. The entropy, conditional entropy and mutual information are described.

In the case of quantum optics applications, when one desires to transmit information using "photons" *it is unrealistic to work with planar harmonic waves, requiring "wave packets" delimited in time and space* [3]. Instead, the impulsive waveforms are used, enabling the study of the temporal behaviour of both perturbation and light stimulated atoms.

Thus, the modeling and simulation succeeds to overcome the previously encountered difficulties as follows: a) to understand the treatment with real and complex signal representation (Fourier) for engineering applications; b)-to solve the equations complicated due to too many qualitative and quantitative approximations; c)- to be in agreement with the software requirements.

2. THE EVOLUTION OF STIMULATED QUANTUM SYSTEMS
Qualitative considerations

We perform a qualitative analysis with C^{∞} class functions, as follows:

- a a stimulated quantum system: $a_t : [t_\alpha, t_\beta] \to \mathcal{A}$
- ζ a perturbing quantum system (e.g. electromagnetic field): $\zeta_{t,\eta} : [t_\alpha, t_\beta] \times \mathcal{R} \to \mathcal{K}$

$$\zeta_{t,\eta} := \begin{cases} \neq \varnothing & t \in (t_0, t_*) \subseteq (t_\alpha, t_\beta) \\ = \varnothing & t \in [t_\alpha, t_\beta] - (t_0, t_*) \end{cases}; \quad \begin{array}{l} \zeta_{t_0,\eta} \equiv \varnothing \equiv \zeta_{t_*,\eta} \\ \lim\limits_{\eta \to 0} \zeta_{t,\eta} = \zeta_{t,0} = \varnothing \end{array}$$

- the Taylor formal series after the control parameter $\eta \in \mathcal{R}$

$$\zeta_{t,\eta} = \zeta_{t,0} + \frac{\eta}{1!} \cdot \left(\frac{\partial \zeta_{t,\eta}}{\partial \eta} \right)_{\eta=0} + \frac{\eta^2}{2!} \cdot \left(\frac{\partial^2 \zeta_{t,\eta}}{\partial \eta^2} \right)_{\eta=0} + \ldots; \ \frac{1}{r!} \cdot \left(\frac{\partial^r \zeta_{t,\eta}}{\partial \eta^r} \right) =: \zeta_{t,\eta}^{<r>}$$

$$\zeta_{t,\eta} = \zeta_{t,0}^{<0>} + \eta \cdot \zeta_{t,0}^{<1>} + \eta^2 \cdot \zeta_{t,0}^{<2>} + \ldots; \ \zeta_{t,\eta}^{<0>} = \zeta_{t,\eta}, \ \ \zeta_{t,0}^{<0>} = \zeta_{t,0} = \varnothing$$

$\zeta_{t,\eta} = \eta \cdot \zeta_{t,0}^{<1>} + \eta^2 \cdot \zeta_{t,0}^{<2>} + \ldots$ so for $\eta \to 0$, asymptotically $\zeta_{t,\eta} \cong \eta \cdot \zeta_{t,0}^{<1>}$

We emphasize the following *algebraic structuring*:

- intensive composition of a disjoint systems $a \in \mathcal{A}$ & $b \in \mathcal{B}$, disjoint $a \cap b = \varnothing = b \cap a$:
  $c = a \oplus b : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ binary composition law, ("*superposition" by interaction*), *intensive in the sense that any event $\omega_{a \oplus b}$ in the compound system $a \oplus b$ is defined by $\omega_{a \oplus b} := \omega_a \wedge \omega_b$, the intensive composition of an event ω_a in the system a with an event ω_b in the system b; we have $a \oplus b = b \oplus a$, $a \oplus \varnothing = \varnothing \oplus a = a$.*
- *hybrid* composition ($\lceil a \rceil$ and $\lfloor \zeta \rfloor$ isolated systems) as a limiting case of *integrated* compound case: $a \,\&\, \zeta = \lim\limits_{a \to \lceil a \rceil, \zeta \to \lfloor \zeta \rfloor} (a \oplus \zeta)$, $\zeta \,\&\, a = \lim\limits_{\zeta \to \lceil \zeta \rceil, a \to \lfloor a \rfloor} (\zeta \oplus a)$.
- absence of perturbation in composition (at $\eta := 0$ or $t := t_0$ or $t := t_*$):

$$(a \oplus \zeta)_{t,0} \equiv (a \oplus \varnothing)_t \equiv a_t \qquad (\zeta \oplus a)_{t,0} \equiv (\varnothing \oplus a)_t \equiv a_t$$
$$(a \oplus \zeta)_{t_0,\eta} \equiv (a \oplus \varnothing)_{t_0} \equiv a_{t_0} \qquad (\zeta \oplus a)_{t_0,\eta} \equiv (\varnothing \oplus a)_{t_0} \equiv a_{t_0}$$
$$(a \oplus \zeta)_{t_*,\eta} \equiv (a \oplus \varnothing)_{t_*} \equiv a_{t_*} \qquad (\zeta \oplus a)_{t_*,\eta} \equiv (\varnothing \oplus a)_{t_*} \equiv a_{t_*}$$

- interaction reduces non-interaction: $\zeta \oplus a \subseteq \zeta \,\&\, a \equiv a \,\&\, \zeta \supseteq a \oplus \zeta$ and we define:

protocol (agreement) subsystems:

$\underleftarrow{a \perp \zeta} \equiv a \perp \zeta = \zeta \,\&\, a - \zeta \oplus a$ ζ by a, $\underleftarrow{\zeta \perp a} \equiv \zeta \perp a := a \,\&\, \zeta - a \oplus \zeta$ a by ζ

$$a \perp \zeta = \varnothing \Leftrightarrow \zeta \oplus a = \zeta \,\&\, a \qquad \zeta \perp a = \varnothing \Leftrightarrow a \oplus \zeta = a \,\&\, \zeta$$
$$(a \perp \zeta)_{t,\eta} = \zeta_{t,\eta} \,\&\, a_t - \zeta_{t,\eta} \oplus a_t \qquad (\zeta \perp a)_{t,\eta} = a_t \,\&\, \zeta_{t,\eta} - a_t \oplus \zeta_{t,\eta}$$

and we have $(a \perp \zeta) = (\zeta \perp a)$, that is the protocol is mutual.

- the protocol is disjunctively filled with the composition

$$(a \perp \zeta) \cap (\zeta \oplus a) = \varnothing, \ (a \perp \zeta) \cup (\zeta \oplus a) = \zeta \,\&\, a, \ \zeta \oplus a = (\zeta \,\&\, a) - (a \perp \zeta)$$
$$(\zeta \perp a) \cap (a \oplus \zeta) = \varnothing, \ (\zeta \perp a) \cup (a \oplus \zeta) = a \,\&\, \zeta, \ a \oplus \zeta = (a \,\&\, \zeta) - (\zeta \perp a)$$

- the absence of perturbation abolishes the protocols:

$(a \perp \zeta)_{t,0} \equiv a_t \perp \varnothing \equiv a_t \,\&\, \varnothing - a_t \oplus \varnothing \equiv a_t - a_t \equiv \varnothing$; $a_t \perp \zeta_{t,0} \equiv \zeta_{t,0} \perp a_t \equiv \varnothing$

$(a \perp \zeta)_{t_0,\eta} \equiv a_{t_0} \perp \varnothing = a_{t_0} \,\&\, \varnothing - a_{t_0} \oplus \varnothing = a_{t_0} - a_{t_0} = \varnothing$

$(a \perp \zeta)_{t_*,\eta} \equiv a_{t_*} \perp \varnothing = a_{t_*} \,\&\, \varnothing - a_{t_*} \oplus \varnothing = a_{t_*} - a_{t_*} = \varnothing$; $a_{t_*} \perp \zeta_{t_*,\eta} \equiv \zeta_{t_*,\eta} \perp a_{t_*} \equiv \varnothing$

- conditioning:

$\underleftarrow{\zeta \,|\, a} \equiv \zeta \,|\, a$, $(\zeta \,|\, a)_{t,\eta} = \zeta_{t,\eta} \,|\, a_t$ the perturbing ζ under conditions *imposed* by a

$\underleftarrow{a \,|\, \zeta} \equiv a \,|\, \zeta$, $(a \,|\, \zeta)_{t,\eta} = a_t \,|\, \zeta_{t,\eta}$ the chosen system a under conditions *imposed* by ζ

"imposed conditions" ≡ "constitutive laws" of the interaction

- the absence of perturbation abolishes conditionalities:

$(a \,|\, \zeta)_{t,0} \equiv a_t \,|\, \varnothing \equiv \varnothing$ $\quad (a \,|\, \zeta)_{t_0,\eta} \equiv a_{t_0} \,|\, \varnothing \equiv \varnothing$ $\quad (a \,|\, \zeta)_{t_*,\eta} \equiv a_{t_*} \,|\, \varnothing \equiv \varnothing$

$(\zeta \,|\, a)_{t,0} \equiv \varnothing \,|\, a_t \equiv \varnothing$ $\quad (\zeta \,|\, a)_{t_0,\eta} \equiv \varnothing \,|\, a_{t_0} \equiv \varnothing$ $\quad (\zeta \,|\, a)_{t_*,\eta} \equiv \varnothing \,|\, a_{t_*} \equiv \varnothing$

- conditionalities are disjunctively complemented by protocols

$(a \,|\, \zeta) \cap (a \perp \zeta) = \varnothing,\; a = (a \,|\, \zeta) \,\&\, (a \perp \zeta),\; a \perp \zeta = a - (a \,|\, \zeta)$

$(\zeta \,|\, a) \cap (\zeta \perp a) = \varnothing,\; \zeta = (\zeta \,|\, a) \,\&\, (\zeta \perp a),\; \zeta \perp a = \zeta - (\zeta \,|\, a)$

- conditionalities determine the compositions (***general relations of interactions***)

$a \oplus \zeta = (a \,\&\, \zeta) - (\zeta \perp a) = a \,\&\, \zeta - [\zeta - (\zeta \,|\, a)] = [(a \,\&\, \zeta) - \zeta] \,\&\, (\zeta \,|\, a) = a \,\&\, (\zeta \,|\, a)$

$\zeta \oplus a = (\zeta \,\&\, a) - (a \perp \zeta) = \zeta \,\&\, a - [a - (a \,|\, \zeta)] = [(\zeta \,\&\, a) - a] \,\&\, (a \,|\, \zeta) = \zeta \,\&\, (a \,|\, \zeta)$

$a \,\&\, (\zeta \,|\, a) = a \oplus \zeta = \zeta \oplus a = \zeta \,\&\, (a \,|\, \zeta)$, $a \,\&\, (\zeta) = a \,\&\, \zeta = \zeta \,\&\, a = \zeta \,\&\, (a)$

These relationships, about the idea of of interaction *system vs. perturbation*, are in agreement with the formal framework of information theory: regarding**:** state vs probability / entropy, protocol vs mutual information / transinformation, interaction vs conditional probabillity / conditional entropy.

3. QUANTITATIVE OPERATOR CONSIDERATIONS

According to quantum physics we have the self-adjoint operators:

- $\boldsymbol{\omega}$ temporal pulse operator ($\boldsymbol{\omega}_{a_t}$, $\boldsymbol{\omega}_{\zeta_{t,\eta}}$, $\boldsymbol{\omega}_{(a\oplus\zeta)_{t,\eta}}$)
- $\mathbf{H}$ hamiltonian operator ($\mathbf{H}_{a_t}$, $\mathbf{H}_{\zeta_{t,\eta}}$, $\mathbf{H}_{(a\oplus\zeta)_{t,\eta}}$)
- the quantum temporal condition (operator format): $\mathbf{H} = \hbar \cdot \boldsymbol{\omega}$
- the total time derivation of an operator $\mathbf{A}$ using the Hamiltonian commutator:
  $d\mathbf{A}/dt = \partial\mathbf{A}/\partial t + [\mathbf{H},\mathbf{A}]$ so that $d\mathbf{H}/dt = \partial\mathbf{H}/\partial t + [\mathbf{H},\mathbf{H}] = \partial\mathbf{H}/\partial t$

The general relationships of the quantitative form of interaction are:

$\oplus$ - integrated composition: $\mathbf{H}_{a_t} + \mathbf{H}_{(\zeta|a)_{t,\eta}} = \mathbf{H}_{(a\oplus\zeta)_{t,\eta}} = \mathbf{H}_{(\zeta\oplus a)_{t,\eta}} = \mathbf{H}_{\zeta_{t,\eta}} + \mathbf{H}_{(a|\zeta)_{t,\eta}}$

& - hybrid composition ($\zeta \,|\, a = \zeta$): $\mathbf{H}_{a_t} + \mathbf{H}_{\zeta_{t,\eta}} = \mathbf{H}_{(a\&\zeta)_{t,\eta}} = \mathbf{H}_{(\zeta\& a)_{t,\eta}} = \mathbf{H}_{\zeta_{t,\eta}} + \mathbf{H}_{a_t}$

For the isolated (unperturbed - time stationary) system a we have $\partial a_t / \partial t = \varnothing$ and the time flows uniformly ($\partial\mathbf{H}_{a_t} / \partial t = \mathbf{0}$), so that $d\mathbf{H}_{a_t} / dt = \partial\mathbf{H}_{a_t} / \partial t = \mathbf{0}$; $\mathbf{H}_{a_t} \equiv \mathbf{H}_a$.

- the equations with eigen states and values are: $\mathbf{H}_a \cdot \mathbf{s}_{\mathrm{n}} = E_n \cdot \mathbf{s}_{\mathrm{n}}$, $\boldsymbol{\omega}_a \cdot \mathbf{s}_{\mathrm{n}} = \omega_n \cdot \mathbf{s}_{\mathrm{n}}$; with $E_n = \hbar \cdot \omega_n$ & spectral differences: $E^{p,m} := E_m - E_p$, $\omega^{p,m} := \omega_m - \omega_p$; $E^{p,m} = \hbar \cdot \omega^{p,m}$.
- the orthonormal basis $\{\mathbf{s}_{\mathrm{n}} \,|\, n \in \mathfrak{I}\}$ of eigenstates has the properties:

$\langle \mathbf{s}_{\mathrm{p}} | \mathbf{s}_{\mathrm{m}} \rangle = \langle \mathbf{s}_{\mathrm{p}} | \cdot \overline{| \mathbf{s}_{\mathrm{m}} \rangle} = \delta_p^m = \overline{\delta_p^m} = \overline{\langle \mathbf{s}_{\mathrm{p}} |} \cdot | \mathbf{s}_{\mathrm{m}} \rangle$, $\langle \mathbf{s}_{\mathrm{p}} | \cdot \overline{| \mathbf{s}_{\mathrm{n}} \rangle} \cdot \overline{\langle \mathbf{s}_{\mathrm{n}} |} \cdot | \mathbf{s}_{\mathrm{m}} \rangle = \delta_p^n \cdot \delta_n^m$,

$\sum_n \langle \mathbf{s}_{\mathrm{p}} | \cdot (\overline{|\mathbf{s}_{\mathrm{n}}\rangle} \cdot \overline{\langle \mathbf{s}_{\mathrm{n}}|}) \cdot |\mathbf{s}_{\mathrm{m}}\rangle = \sum_n \delta_p^n \cdot \delta_n^m$, $\sum_n \overline{|\mathbf{s}_{\mathrm{n}}\rangle} \cdot \overline{\langle \mathbf{s}_{\mathrm{n}}|} = [\mathbf{I}] = \sum_n |\mathbf{s}_{\mathrm{n}}\rangle \cdot \langle \mathbf{s}_{\mathrm{n}}|$

- the quantum state of (isolated) a_{t}, in general format, is a linear combinations $\boldsymbol{\psi}_{a_t} = \sum_n c_{a_t}^n \cdot \mathbf{s}_{\mathrm{n}}$ & $c_{a_t}^n / c_{a_{t_o}}^n = e^{-i\cdot\omega_n\cdot(t-t_o)}$ (*Schrödinger eq.*) hence $\boldsymbol{\psi}_{a_t} = \sum_n c_{a_{t_o}}^n \cdot e^{-i\cdot\omega_n\cdot(t-t_o)} \cdot \mathbf{s}_{\mathrm{n}}$ where usually $t_o := t_0 := 0$.

We analyze the state for hybrid system $a \& \zeta = \zeta \& a$ vs. for composed system $a \oplus \zeta$; so $\mathbf{H}_a + \mathbf{H}_\zeta = \mathbf{H}_{a\&\zeta} = \mathbf{H}_{\zeta\&a} = \mathbf{H}_\zeta + \mathbf{H}_a$ and we have two formats:

- *format*: $a \oplus \zeta = a \& \zeta - (\zeta \perp a) = \zeta \& a - (a \perp \zeta) = \zeta \oplus a$

$\mathbf{H}_a + \mathbf{H}_\zeta - \mathbf{H}_{\zeta\perp a} = \mathbf{H}_{a\oplus\zeta} = \mathbf{H}_{\zeta\oplus a} = \mathbf{H}_\zeta + \mathbf{H}_a - \mathbf{H}_{a\perp\zeta}$

$\mathbf{H}_{\zeta\perp a} := \mathbf{H}_a + \mathbf{H}_\zeta - \mathbf{H}_{a\oplus\zeta}$ $\quad \mathbf{H}_{a\perp\zeta} := \mathbf{H}_\zeta + \mathbf{H}_a - \mathbf{H}_{\zeta\oplus a}$

$\mathbf{H}_a + \mathbf{H}_\zeta = \mathbf{H}_{a\oplus\zeta} = \mathbf{H}_{\zeta\oplus a} = \mathbf{H}_\zeta + \mathbf{H}_a \Leftrightarrow \mathbf{H}_{\zeta\perp a} = \mathbf{0} = \mathbf{H}_{a\perp\zeta}$,

- *format*: $a \& (\zeta | a) = a \oplus \zeta = \zeta \oplus a = \zeta \& (a | \zeta)$, $\mathbf{H}_a + \mathbf{H}_{\zeta|a} = \mathbf{H}_{a\oplus\zeta} = \mathbf{H}_{\zeta\oplus a} = \mathbf{H}_\zeta + \mathbf{H}_{a|\zeta}$

$\mathbf{H}_{\zeta|a} := \mathbf{H}_{a\oplus\zeta} - \mathbf{H}_a$ (perturbant hamiltonian) $\mathbf{H}_{a|\zeta} := \mathbf{H}_{\zeta\oplus a} - \mathbf{H}_\zeta$ (perturbed hamiltonian)

$\mathbf{H}_a + \mathbf{H}_\zeta = \mathbf{H}_{a\oplus\zeta} = \mathbf{H}_{\zeta\oplus a} = \mathbf{H}_\zeta + \mathbf{H}_a \Leftrightarrow \zeta | a \equiv \zeta$ & $a | \zeta \equiv a$

We describe perturbing action in the format $\mathbf{H}_{a\oplus\zeta} = \mathbf{H}_a + \mathbf{H}_{\zeta|a}$, as follows:

- the quantum system state $a \oplus \zeta$ (general format) is $\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}} = \sum_n c_{a_t\oplus\zeta_{t,\eta}}^n \cdot \mathbf{s}_{\mathrm{n}}$ where $c_{a_t\oplus\zeta_{t,\eta}}^n \neq c_{a_t}^n$, because a interacts with ζ; $c_{a_t\oplus\zeta_{t,0}}^n = c_{a_t}^n$, $\boldsymbol{\psi}_{a_t\oplus\zeta_{t,0}} \equiv \boldsymbol{\psi}_{a_t}$, $c_{a_{t_0}\oplus\zeta_{t_0,\eta}}^n = c_{a_{t_0}}^n$, $\boldsymbol{\psi}_{a_{t_0}\oplus\zeta_{t_0,\eta}} \equiv \boldsymbol{\psi}_{a_{t_0}}$, $\boldsymbol{\psi}_{a_{t_0}\oplus\zeta_{t_0,\eta}} = \sum_n c_{a_{t_0}\oplus\zeta_{t_0,\eta}}^n \cdot \mathbf{s}_{\mathrm{n}}$.

- we practice a double coefficient relativization $\dfrac{c_{a_t\oplus\zeta_{t,\eta}}^n}{c_{a_t}^n / c_{a_{t_0}}^n} = \dfrac{c_{a_t\oplus\zeta_{t,\eta}}^n}{e^{-i\cdot\omega_n\cdot(t-t_0)}} =: \gamma_{a_t\oplus\zeta_{t,\eta}}^n$ so that:

$\gamma_{a_t\oplus\zeta_{t,\eta}}^n := c_{a_t\oplus\zeta_{t,\eta}}^n \cdot e^{i\cdot\omega_n\cdot(t-t_0)}$, $c_{a_t\oplus\zeta_{t,\eta}}^n = \gamma_{a_t\oplus\zeta_{t,\eta}}^n \cdot e^{-i\cdot\omega_n\cdot(t-t_0)}$, $\gamma_{a_t\oplus\zeta_{t,0}}^n = c_{a_t\oplus\zeta_{t,0}}^n \cdot e^{i\cdot\omega_n\cdot(t-t_0)}$,

$\gamma_{a_t}^n := c_{a_t}^n \cdot e^{i\cdot\omega_n\cdot(t-t_0)}$, $\gamma_{a_{t_0}}^n = c_{a_{t_0}}^n$, $c_{a_t\oplus\zeta_{t,0}}^n = \gamma_{a_t\oplus\zeta_{t,0}}^n \cdot e^{-i\cdot\omega_n\cdot(t-t_0)}$, $c_{a_t}^n = \gamma_{a_t}^n \cdot e^{-i\cdot\omega_n\cdot(t-t_0)}$,

$c_{a_{t_0}}^n = \gamma_{a_{t_0}}^n$, $\gamma_{a_{t_0}\oplus\zeta_{t_0,\eta}}^n = c_{a_{t_0}\oplus\zeta_{t_0,\eta}}^n = c_{a_{t_0}}^n = \gamma_{a_{t_0}}^n$

- the state vectors are:

$\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}} = \sum_n c_{a_t\oplus\zeta_{t,\eta}}^n \cdot \mathbf{s}_{\mathrm{n}} = \sum_n \gamma_{a_t\oplus\zeta_{t,\eta}}^n \cdot e^{-i\cdot\omega_n\cdot(t-t_0)} \cdot \mathbf{s}_{\mathrm{n}}$, $\boldsymbol{\psi}_{a_{t_0}} = \sum_n c_{a_{t_0}}^n \cdot \mathbf{s}_{\mathrm{n}} = \sum_n \gamma_{a_{t_0}}^n \cdot \mathbf{s}_{\mathrm{n}}$ where $\boldsymbol{\psi}_{a_{t_0}\oplus\zeta_{t_0,\eta}} \equiv \boldsymbol{\psi}_{a_{t_0}\oplus\varnothing} = \boldsymbol{\psi}_{a_{t_0}}$

- If $\boldsymbol{\psi}_{a_{t_0}}^p = \sum_n c_{a_{t_0}}^{n|p} \cdot \mathbf{s}_{\mathrm{n}} \equiv \sum_n c_{a_{t_0}\oplus\zeta_{t_0,\eta}}^{n|p} \cdot \mathbf{s}_{\mathrm{n}}$ then $\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^p = \sum_{n'} c_{a_t\oplus\zeta_{t,\eta}}^{n'|p} \cdot \mathbf{s}_{\mathrm{n'}}$, and the probability $P_{\boldsymbol{\psi}_{a_{t_0}}^p \mapsto \boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^m} \equiv P_{\eta,t|t_0}^{m|p}$ to find a in the state $\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^m = \sum_{n''} c_{a_t\oplus\zeta_{t,\eta}}^{n''|m} \cdot \mathbf{s}_{\mathrm{n''}}$ represent the probability that the system a to perform in the temporal interval $[t_0, t]$ transition from state $\boldsymbol{\psi}_{a_{t_0}}^p$ in the state $\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^m$ or otherwise, from state $\boldsymbol{\psi}_{a_{t_0}\oplus\zeta_{t_0,\eta}}^p$ in the state $\boldsymbol{\psi}_{a_{t\geq t_0}\oplus\zeta_{t\geq t_0,\eta}}^m$ is:

$$P_{\eta,t|t_0}^{m|p}=P_{\boldsymbol{\psi}_{a_{t_0}}^{p}\mapsto\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^{m}}=\left|\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^{p}\middle|\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}^{m}\right\rangle\right|^2=\left|\left\langle\sum_{n'}c_{a_t\oplus\zeta_{t,\eta}}^{n'|p}\cdot\mathbf{s}_{n'}\middle|\sum_{n''}c_{a_t\oplus\zeta_{t,\eta}}^{n''|m}\cdot\mathbf{s}_{n''}\right\rangle\right|^2$$

$$=\left|\sum_{n'}\sum_{n''}c_{a_t\oplus\zeta_{t,\eta}}^{n'|p}\cdot\overline{c_{a_t\oplus\zeta_{t,\eta}}^{n''|m}}\cdot\langle\mathbf{s}_{n'}|\mathbf{s}_{n''}\rangle\right|^2=\left|\sum_{n'}\sum_{n''}c_{a_t\oplus\zeta_{t,\eta}}^{n'|p}\cdot\overline{c_{a_t\oplus\zeta_{t,\eta}}^{n''|m}}\cdot\delta_{n'}^{n''}\right|^2=\left|\sum_{n}c_{a_t\oplus\zeta_{t,\eta}}^{n|p}\cdot\overline{c_{a_t\oplus\zeta_{t,\eta}}^{n|m}}\right|^2$$

We have also: $P_{\eta,t|t_0}^{m|p}=\left|\sum_n\gamma_{a_t\oplus\zeta_{t,\eta}}^{n|p}\cdot e^{-i\cdot\omega_n\cdot(t-t_0)}\cdot\overline{\gamma_{a_t\oplus\zeta_{t,\eta}}^{n|m}\cdot e^{-i\cdot\omega_n\cdot(t-t_0)}}\right|^2=\left|\sum_n\gamma_{a_t\oplus\zeta_{t,\eta}}^{n|p}\cdot\overline{\gamma_{a_t\oplus\zeta_{t,\eta}}^{n|m}}\right|^2$

We proceed to perturbation series expansion of interaction hamiltonian:

- $\mathbf{H}_{\zeta_{t,\eta}|a_t}=\mathbf{H}_{\zeta_{t,0}|a_t}+\frac{\eta}{1!}\cdot\left(\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}\right)_{\eta:=0}+\frac{\eta^2}{2!}\cdot\left(\frac{\partial^2\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta^2}\right)_{\eta:=0}+\ldots$
- but $\forall t$, $\lim_{\eta\to0}\mathbf{H}_{\zeta_{t,\eta}|a_t}=\mathbf{H}_{\zeta_{t,0}|a_t}=\mathbf{H}_{\varnothing|a_t}=\mathbf{H}_{\varnothing}=\mathbf{0}$, so
- $\mathbf{H}_{\zeta_{t,\eta}|a_t}=\frac{\eta}{1!}\cdot\left(\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}\right)_{\eta:=0}+\frac{\eta^2}{2!}\cdot\left(\frac{\partial^2\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta^2}\right)_{\eta:=0}+\ldots\equiv\eta\cdot\mathcal{H}_{\zeta_{t,\eta}|a_t}$
- $\mathcal{H}_{\zeta_{t,\eta}|a_t}\coloneqq\mathbf{H}_{\zeta_{t,\eta}|a_t}/\eta$ is the parametric Hamiltonian perturbation mean density:

$$\mathcal{H}_{\zeta_{t,\eta}|a_t}\equiv\mathcal{H}_{\zeta_{t,0}|a_t}+\eta\cdot\mathfrak{R}_{\zeta_{t,\eta}|a_t},\ \mathcal{H}_{\zeta_{t,0}|a_t}\coloneqq\left(\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}\right)_{\eta:=0},\ \mathbf{R}_{\zeta_{t,\eta}|a_t}\equiv\sum_{r=2}^{\infty}\frac{\eta^{r-2}}{r!}\left(\frac{\partial^r\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta^r}\right)_{\eta:=0}$$

$$\mathcal{H}_{\zeta_{t,0}|a_t}=\lim_{\eta\to0}\mathcal{H}_{\zeta_{t,\eta}|a_t}=\lim_{\eta\to0}\frac{\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\eta}=\left(\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}\right)_{\eta:=0}\neq\mathbf{0}$$

$\mathbf{H}_{\zeta_{t,\eta}|a_t}\equiv\eta\cdot\mathcal{H}_{\zeta_{t,0}|a_t}+O(\eta^2)$, for $\eta\to0$, asymptotic: $\mathbf{H}_{\zeta_{t,\eta}|a_t}\cong\eta\cdot\mathcal{H}_{\zeta_{t,0}|a_t}$

$\mathbf{H}_{\zeta_{t,\eta}\oplus a_t}=\mathbf{H}_{a_t}+\mathbf{H}_{\zeta_{t,\eta}|a_t}\equiv\mathbf{H}_{a_t}+\eta\cdot\mathcal{H}_{\zeta_{t,\eta}|a_t}\cong\mathbf{H}_{a_t}+\eta\cdot\mathcal{H}_{\zeta_{t,0}|a_t}$

- The Schrödinger equation [2] for the system is:

$i\cdot\hbar\cdot\frac{d\boldsymbol{\psi}(t)}{dt}=\mathbf{H}\cdot\boldsymbol{\psi}(\mathrm{t})$ or in matriceal form: $i\cdot\hbar\cdot\frac{\partial}{\partial t}|\boldsymbol{\psi}(\mathrm{t})\rangle=[\mathbf{H}]\cdot|\boldsymbol{\psi}(\mathrm{t})\rangle$ and we apply:

$$i\cdot\hbar\cdot\frac{\partial\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}}{\partial t}=\mathbf{H}_{a_t\oplus\zeta_{t,\eta}}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}=\mathbf{H}_{a_t}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}+\eta\cdot\mathcal{H}_{\zeta_{t,0}|a_t}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}},$$

$$i\cdot\hbar\cdot\frac{\partial}{\partial t}\left|\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\right\rangle=[\mathbf{H}_{a_t\oplus\zeta_{t,\eta}}]\cdot\left|\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\right\rangle,\ i\cdot\hbar\cdot\frac{\partial}{\partial t}\left|\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\right\rangle=[\mathbf{H}_{a_t\oplus\zeta_{t,\eta}}]\cdot\left|\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\right\rangle$$

we project this equation on an eigenstate $\mathbf{s}_{\mathrm{m}}$ of $\mathbf{H}_{a_t}\equiv\mathbf{H}_a$:

$i\cdot\hbar\cdot\frac{\partial\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle}{\partial t}=\left\langle\mathbf{H}_{a_t}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle+\eta\cdot\left\langle\mathcal{H}_{\zeta_{t,0}|a_t}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle$ with:

$$\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}=\sum_n c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\mathbf{s}_{\mathrm{n}}=\sum_n\gamma_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot e^{-\mathrm{i}\cdot\omega_n\cdot(t-t_0)}\cdot\mathbf{s}_{\mathrm{n}}$$

$$\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle=\left\langle\sum_n c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\mathbf{s}_{\mathrm{n}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle=\sum_n c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\langle\mathbf{s}_{\mathrm{n}}|\mathbf{s}_{\mathrm{m}}\rangle=\sum_n c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\delta_n^m=c_{a_t\oplus\zeta_{t,\eta}}^{m}$$

$$\left\langle\mathbf{H}_{a_t}\cdot\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle=\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{H}_{a_t}^{\dagger}\cdot\mathbf{s}_{\mathrm{m}}\right\rangle=\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|\mathbf{H}_{a_t}\cdot\mathbf{s}_{\mathrm{m}}\right\rangle=\left\langle\boldsymbol{\psi}_{a_t\oplus\zeta_{t,\eta}}\middle|E_m\cdot\mathbf{s}_{\mathrm{m}}\right\rangle$$

$$=E_m\cdot\left\langle\sum_n c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\mathbf{s}_{\mathrm{n}}\middle|\mathbf{s}_{\mathrm{m}}\right\rangle=E_m\cdot\sum_{\mathrm{n}}c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\langle\mathbf{s}_{\mathrm{n}}|\mathbf{s}_{\mathrm{m}}\rangle=E_m\cdot\sum_{\mathrm{n}}c_{a_t\oplus\zeta_{t,\eta}}^{n}\cdot\delta_n^m=E_m\cdot c_{a_t\oplus\zeta_{t,\eta}}^{m}$$

$$\left\langle \mathcal{H}_{\zeta_{t,0}|a_t} \cdot \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \mathbf{s}_\mathrm{m} \right\rangle = \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \mathcal{H}^\dagger_{\zeta_{t,0}|a_t} \cdot \mathbf{s}_\mathrm{m} \right\rangle = \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \mathcal{H}_{\zeta_{t,0}|a_t} \cdot \mathbf{s}_\mathrm{m} \right\rangle$$
$$= \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \cdot \overline{[\mathcal{H}_{\zeta_{t,0}|a_t}]} \cdot \overline{|\mathbf{s}_\mathrm{m}\rangle} = \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \cdot [\mathbf{I}] \cdot \overline{[\mathcal{H}_{\zeta_{t,0}|a_t}]} \cdot \overline{|\mathbf{s}_\mathrm{m}\rangle}$$
$$= \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \cdot (\sum_n \overline{|\mathbf{s}_\mathrm{n}\rangle} \cdot \overline{\langle \mathbf{s}_\mathrm{n}|}) \cdot \overline{[\mathcal{H}_{\zeta_{t,0}|a_t}]} \cdot \overline{|\mathbf{s}_\mathrm{m}\rangle}$$
$$= \sum_n \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \cdot \overline{|\mathbf{s}_\mathrm{n}\rangle} \cdot \overline{\langle \mathbf{s}_\mathrm{n}|} \cdot \overline{[\mathcal{H}_{\zeta_{t,0}|a_t}]} \cdot \overline{|\mathbf{s}_\mathrm{m}\rangle} = \sum_n \left\langle \boldsymbol{\psi}_{a_t \oplus \zeta_{t,\eta}} \middle| \mathbf{s}_\mathrm{n} \right\rangle \cdot \overline{\langle \mathbf{s}_\mathrm{n} | \cdot [\mathcal{H}_{\zeta_{t,0}|a_t}] \cdot |\mathbf{s}_\mathrm{m}\rangle}$$

and with the notation $\langle \mathbf{s}_\mathrm{n} | \cdot [\mathcal{H}_{\zeta_{t,0}|a_t}] \cdot |\mathbf{s}_\mathrm{m}\rangle = \mathcal{H}^{n,m}(t)$, after some elementary calculations in Schrödinger equation we obtain:

$$i \cdot \hbar \cdot \frac{\partial \gamma^m_{a_t \oplus \zeta_{t,\eta}}}{\partial t} = \eta \cdot \sum_n \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)} \cdot \gamma^n_{a_t \oplus \zeta_{t,\eta}}$$

In this equation we develop $\gamma^n_{a_t \oplus \zeta_{t,\eta}}$ as a power series in η,

$$\gamma^n_{a_t \oplus \zeta_{t,\eta}} = \gamma^n_{a_t \oplus \zeta_{t,0}} + \frac{\eta}{1!} \cdot \left(\frac{\partial \gamma^n_{a_t \oplus \zeta_{t,\eta}}}{\partial \eta} \right)_{\eta:=0} + \frac{\eta^2}{2!} \cdot \left(\frac{\partial^2 \gamma^n_{a_t \oplus \zeta_{t,\eta}}}{\partial \eta^2} \right)_{\eta:=0} + \ldots,\ \varsigma^n_{t,r} := \frac{1}{r!} \cdot \left(\frac{\partial^r \gamma^n_{a_t \oplus \zeta_{t,\eta}}}{\partial \eta^r} \right)_{\eta:=0}$$

$$\varsigma^n_{t,0} := \gamma^n_{a_t \oplus \zeta_{t,0}} \equiv \gamma^n_{a_t},\ \gamma^n_{a_t \oplus \zeta_{t,\eta}} = \varsigma^n_{t,0} + \eta \cdot \varsigma^n_{t,1} + \eta^2 \cdot \varsigma^n_{t,2} + \ldots,\ \varsigma^n_{t,0} = \gamma^n_{a_t},\ \varsigma^n_{t_0,0} = \gamma^n_{a_{t_0}} = c^n_{a_{t_0}}$$
$$\gamma^m_{a_t \oplus \zeta_{t,\eta}} = \varsigma^m_{t,0} + \eta \cdot \varsigma^m_{t,1} + \eta^2 \cdot \varsigma^m_{t,2} + \ldots,\ \varsigma^m_{t,0} = \gamma^m_{a_t},\ \varsigma^m_{t_0,0} = \gamma^m_{a_{t_0}} = c^m_{a_{t_0}},\ \gamma^n_{a_{t_0} \oplus \zeta_{t_0,\eta}} \equiv c^n_{a_{t_0} \oplus \zeta_{t_0,\eta}} \equiv c^n_{a_{t_0}}$$
$$\gamma^n_{a_{t_0} \oplus \zeta_{t_0,\eta}} \equiv \varsigma^n_{t_0,0} + \eta \cdot \varsigma^n_{t_0,1} + \eta^2 \cdot \varsigma^n_{t_0,2} + \ldots,\ c^n_{a_{t_0}} \equiv c^n_{a_{t_0}} + \eta \cdot \varsigma^n_{t_0,1} + \eta^2 \cdot \varsigma^n_{t_0,2} + \ldots$$
$$0 \equiv \eta \cdot \varsigma^n_{t_0,1} + \eta^2 \cdot \varsigma^n_{t_0,2} + \ldots,\ 0 = \varsigma^n_{t_0,1} = \varsigma^n_{t_0,2} = \ldots$$
$$i \cdot \hbar \cdot \frac{\partial}{\partial t}[\varsigma^m_{t,0} + \eta \cdot \varsigma^m_{t,1} + \eta^2 \cdot \varsigma^m_{t,2} + ..] = \eta \cdot \sum_n \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)} \cdot [\varsigma^n_{t_0,0} + \eta \cdot \varsigma^n_{t_0,1} + \eta^2 \cdot \varsigma^n_{t_0,2} + ..]$$

and by identifying the coefficients, we obtain the system of differential equations:

$$i \cdot \hbar \cdot \frac{d\varsigma^m_{t,0}}{dt} = 0,\ i \cdot \hbar \cdot \frac{d\varsigma^m_{t,1}}{dt} = \sum_n \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)} \cdot \varsigma^n_{t,0},$$
$$i \cdot \hbar \cdot \frac{d\varsigma^m_{t,2}}{dt} = \sum_n \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)} \cdot \varsigma^n_{t,1},\ \ldots,\ i \cdot \hbar \cdot \frac{d\varsigma^m_{t,r}}{dt} = \sum_n \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)} \cdot \varsigma^n_{t,r-1}$$

which can be solved *iteratively*.

4. TRANSITION PROBABILITY

We consider the possibility of transitions $s_p \mapsto s_m$ (type $m|p$) with probability $P^{m|p}_{\eta,t|t_0} = |c^{m|p}_{a_t \oplus \zeta_{t,\eta}}|^2$, where $c^{n|p}_{a_t \oplus \zeta_{t,\eta}} = \gamma^{n|p}_{a_t \oplus \zeta_{t,\eta}} \cdot e^{-i\cdot\omega_n\cdot(t-t_0)}$ and:

- $\gamma^{n|p}_{a_t \oplus \zeta_{t,\eta}} = \varsigma^{n|p}_{t,0} + \eta \cdot \varsigma^{n|p}_{t,1} + \eta^2 \cdot \varsigma^{n|p}_{t,2} + \ldots,\ \varsigma^{n|p}_{t,0} = \gamma^{n|p}_{a_t},\ \varsigma^{n|p}_{t_0,0} = \gamma^{n|p}_{a_{t_0}} = c^{n|p}_{a_{t_0}} = \delta^n_p,$
- $\gamma^{n|p}_{a_{t_0} \oplus \zeta_{t_0,\eta}} \equiv \varsigma^{n|p}_{t_0,0} + \eta \cdot \varsigma^{n|p}_{t_0,1} + \eta^2 \cdot \varsigma^{n|p}_{t_0,2} + \ldots,\ \gamma^{n|p}_{a_{t_0}} \equiv \varsigma^{n|p}_{t_0,0} + \eta \cdot \varsigma^{n|p}_{t_0,1} + \eta^2 \cdot \varsigma^{n|p}_{t_0,2} + \ldots,$
- $\gamma^{n|p}_{a_{t_0}} \equiv \gamma^{n|p}_{a_{t_0}} + \eta \cdot \varsigma^{n|p}_{t_0,1} + \eta^2 \cdot \varsigma^{n|p}_{t_0,2} + \ldots,\ 0 \equiv \eta \cdot \varsigma^{n|p}_{t_0,1} + \eta^2 \cdot \varsigma^{n|p}_{t_0,2} + \ldots,\ 0 = \varsigma^{n|p}_{t_0|1} = \varsigma^{n|p}_{t_0|2} = \ldots$
- $c^{n|p}_{a_t \oplus \zeta_{t,\eta}} = (\varsigma^{n|p}_{t,0} + \eta \cdot \varsigma^{n|p}_{t,1} + \eta^2 \cdot \varsigma^{n|p}_{t,2} + \ldots) \cdot e^{-i\cdot\omega_n\cdot(t-t_0)}$
- $c^{m|p}_{a_t \oplus \zeta_{t,\eta}} = (\varsigma^{m|p}_{t,0} + \eta \cdot \varsigma^{m|p}_{t,1} + \eta^2 \cdot \varsigma^{m|p}_{t,2} + \ldots) \cdot e^{-i\cdot\omega_m\cdot(t-t_0)}$

- $i\cdot\hbar\cdot\dfrac{d\varsigma_{t,0}^{m|p}}{dt}=0 \Rightarrow \forall t\,,\ \varsigma_{t,0}^{m|p}=\varsigma_{t_0,0}^{m|p}=\gamma_{a_{t_0}}^{m|p}=c_{a_{t_0}}^{m|p}=\delta_p^m\,,\ \ \varsigma_{t,0}^{n|p}=\delta_p^n$

for $m\neq p$ and $0\neq\eta\to 0$ we have $c_{a_t\oplus\zeta_{t,\eta}}^{m|p}\cong\eta\cdot\varsigma_{t,1}^{m|p}\cdot e^{-i\cdot\omega_m\cdot(t-t_0)}$,

$$i\cdot\hbar\cdot\frac{d\varsigma_{t,1}^{m|p}}{dt}=\sum_n\overline{\mathcal{H}^{n,m}(t)}\cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)}\cdot\varsigma_{t,0}^{n|p}\,,\ i\cdot\hbar\cdot\frac{d\varsigma_{t,1}^{m|p}}{dt}=\sum_n\overline{\mathcal{H}^{n,m}(t)}\cdot e^{i\cdot\omega^{n,m}\cdot(t-t_0)}\cdot\delta_p^n\,,$$

$\dfrac{d\varsigma_{t,1}^{m|p}}{dt}=\dfrac{e^{-i\cdot\omega^{p,m}\cdot t_0}}{i\cdot\hbar}\cdot\overline{\mathcal{H}^{p,m}(t)}\cdot e^{i\cdot\omega^{p,m}\cdot t}$ and by integration $\int_{t_0}^{t}\ldots\cdot dt'$ with $t_0\le t\le t_*$ result:

$\varsigma_{t,1}^{m|p}-\varsigma_{t_0,1}^{m|p}=\dfrac{e^{-i\cdot\omega^{p,m}\cdot t_0}}{i\cdot\hbar}\cdot\int_{t_0}^{t}\overline{\mathcal{H}^{p,m}(t')}\cdot e^{i\cdot\omega^{p,m}\cdot t'}\cdot dt'$ with $0=\varsigma_{t_0,1}^{m|p}$ and $\omega^{p,m}=\omega_m-\omega_p$ so:

$\varsigma_{t,1}^{m|p}=\dfrac{e^{-i\cdot\omega^{p,m}\cdot t_0}}{i\cdot\hbar}\cdot\int_{t_0}^{t}\overline{\mathcal{H}^{p,m}(t')}\cdot e^{i\cdot\omega^{p,m}\cdot t'}\cdot dt'$; with: $c_{a_t\oplus\zeta_{t,\eta}}^{m|p}\cong\eta\cdot\varsigma_{t,1}^{m|p}\cdot e^{-i\cdot\omega_m\cdot(t-t_0)}$ we have:

$c_{a_t\oplus\zeta_{t,\eta}}^{m|p}\cong\dfrac{\eta\cdot e^{-i\cdot(\omega_m\cdot t-\omega_p\cdot t_0)}}{i\cdot\hbar}\cdot\int_{t_0}^{t}\overline{\mathcal{H}^{p,m}(t')\cdot e^{-i\cdot\omega^{p,m}\cdot t'}}\cdot dt'$ and the transition probability is:

$$P_{\eta,t|t_0}^{m|p}=|c_{a_t\oplus\zeta_{t,\eta}}^{m|p}|^2\cong\left(\frac{\eta}{\hbar}\right)^2\cdot\left|\int_{t_0}^{t}\overline{\mathcal{H}^{p,m}(t')}\cdot e^{-i\cdot\omega^{p,m}\cdot t'}\cdot dt'\right|^2$$

For the entire duration of the perturbation ($t:=t_*=t_0+T$):

$$c_{a_{t_*}\oplus\zeta_{t_*,\eta}}^{m|p}\cong\frac{\eta\cdot e^{-i\cdot(\omega_m\cdot t_*-\omega_p\cdot t_0)}}{i\cdot\hbar}\cdot\int_{t_0}^{t_*}\overline{\mathcal{H}^{p,m}(t)}\cdot e^{-i\cdot\omega^{p,m}\cdot t}\cdot dt$$

$$P_{\eta,t_*|t_0}^{m|p}=|c_{a_{t_*}\oplus\zeta_{t_*,\eta}}^{m|p}|^2\cong\left(\frac{\eta}{\hbar}\right)^2\cdot\left|\int_{t_0}^{t_*}\overline{\mathcal{H}^{p,m}(t)}\cdot e^{-i\cdot\omega^{p,m}\cdot t}\cdot dt\right|^2$$

Introducing the complex representation:

$\underline{\mathcal{H}}_{t_0,t_*}^{p,m}(\omega):=\int_{t_0}^{t_*}\mathcal{H}^{p,m}(t)\cdot e^{-i\cdot\omega\cdot t}\cdot dt\,,\qquad \overline{\underline{\mathcal{H}}_{t_0,t_*}^{p,m}(-\omega)}:=\int_{t_0}^{t_*}\overline{\mathcal{H}^{p,m}(t)}\cdot e^{-i\cdot\omega\cdot t}\cdot dt$ we have:

$$c_{a_{t_*}\oplus\zeta_{t_*,\eta}}^{m|p}\cong\frac{\eta\cdot e^{-i\cdot(\omega_m\cdot t_*-\omega_p\cdot t_0)}}{i\cdot\hbar}\cdot\overline{\underline{\mathcal{H}}_{t_0,t_*}^{p,m}(-\omega^{p,m})}\quad P_{\eta,t_*|t_0}^{m|p}=|c_{a_{t_*}\oplus\zeta_{t_*,\eta}}^{m|p}|^2\cong\left(\frac{\eta}{\hbar}\right)^2\cdot|\,\underline{\mathcal{H}}_{t_0,t_*}^{p,m}(-\omega^{p,m})\,|^2$$

5. TEMPORAL PERTURBATION – MONOPULS

In this case the perturbant hamiltonian is $\mathbf{H}_{\zeta_{t,\eta}|a_t}:=\eta\cdot\mathcal{W}\cdot h(t)$ where $\mathcal{W}$ is a structural (atemporal) operator, and $h(t):=\begin{cases}u(\mathrm{t}) & t\in(t_0,t_*)\subseteq(t_\alpha,t_\beta)\\ 0 & t\in[t_\alpha,t_\beta]-(t_0,t_*)\end{cases}$ with a temporal form factor $|u(t)|\le 1$; we define: $\varphi_t\equiv\varphi(t):=Arccos[u(t)]$ the temporal phase of perturbation, so $u(t):=\cos(\varphi_t)$, $\theta_u:=\varphi_{t_*}-\varphi_{t_0}=\omega_0\cdot T$; $T:=t_*-t_0$ is the duration of perturbation. Particular forms of general relations are:

$$\mathcal{H}_{\zeta_{t,0}|a_t}=\left(\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}\right)_{\eta:=0}=\frac{\partial\mathbf{H}_{\zeta_{t,\eta}|a_t}}{\partial\eta}=\mathcal{W}\cdot h(t)\quad\left\langle\mathbf{s}_{\mathrm{p}}\right|\cdot[\mathcal{W}]\cdot\left|\mathbf{s}_{\mathrm{m}}\right\rangle=:\mathcal{W}^{p,m}\equiv\hbar\cdot w^{p,m}$$

$$\mathcal{H}^{p,m}(t)=\left\langle \mathbf{s}_{\mathrm{p}}\right|\cdot[\mathcal{H}_{\zeta_{t,0}|a_t}]\cdot\left|\mathbf{s}_{\mathrm{m}}\right\rangle=\left\langle \mathbf{s}_{\mathrm{p}}\right|\cdot[\mathcal{W}]\cdot\left|\mathbf{s}_{\mathrm{m}}\right\rangle\cdot h(t)=\mathcal{W}^{p,m}\cdot h(t)=\hbar\cdot w^{p,m}\cdot h(t)$$

$$\underline{\mathcal{H}}^{p,m}_{t_0,t_*}(\omega)=\hbar\cdot w^{p,m}\cdot\int_{t_0}^{t_*}u(t)\cdot e^{-i\cdot\omega\cdot t}\cdot dt=\hbar\cdot w^{p,m}\cdot\underline{u}_{t_0,t_*}(\omega),\ \underline{u}_{t_0,t_*}(\omega):=\int_{t_0}^{t_*}u(t)\cdot e^{-i\cdot\omega\cdot t}\cdot dt$$

$$c^{m|p}_{a_t\oplus\zeta_{t,\eta}}\cong\frac{\eta\cdot e^{-i\cdot(\omega_m\cdot t-\omega_p\cdot t_0)}}{i\cdot\hbar}\cdot\int_{t_0}^{t}\overline{\mathcal{H}^{p,m}(t')}\cdot e^{-i\cdot\omega^{p,m}\cdot t'}\cdot dt'=-i\cdot\eta\cdot e^{-i\cdot(\omega_m\cdot t-\omega_p\cdot t_0)}\cdot\overline{w^{p,m}}\cdot\underline{u}_{t_0,t}(\omega^{p,m})$$

the transition probability is:

$P^{m|p}_{\eta,t|t_0}=|c^{m|p}_{a_t\oplus\zeta_{t,\eta}}|^2=|\eta\cdot w^{p,m}|^2\cdot|\underline{u}_{t_0,t}(\omega^{p,m})|^2,\ P^{m|p}_{\eta,t_*|t_0}\cong|\eta\cdot w^{p,m}|^2\cdot|\underline{u}_{t_0,t_*}(\omega^{p,m})|^2$; we calculate

$$\underline{u}_{t_0,t_*}(\omega)=\frac{e^{-i\cdot\omega\cdot t_0}}{2\cdot i}\cdot\left\{\left[\frac{e^{i\cdot(\theta_u-\theta)}}{\varphi'_{t_*}-\omega}-\frac{1}{\varphi'_{t_0}-\omega}\right]\cdot e^{i\cdot\varphi_{t_0}}-\left[\frac{e^{-i\cdot(\theta_u+\theta)}}{\varphi'_{t_*}+\omega}-\frac{1}{\varphi'_{t_0}+\omega}\right]\cdot e^{-i\cdot\varphi_{t_0}}\right\}\tag{1}$$

for: $\varphi_0:=\pi/2$ result $\underline{u}_{t_0,t_*}(\omega)=\frac{e^{-i\cdot\omega\cdot t_0}}{2}\cdot\left\{\left[\frac{e^{i\cdot(\theta_u-\theta)}}{\varphi'_{t_*}-\omega}-\frac{1}{\varphi'_{t_0}-\omega}\right]+\left[\frac{e^{-i\cdot(\theta_u+\theta)}}{\varphi'_{t_*}+\omega}-\frac{1}{\varphi'_{t_0}+\omega}\right]\right\}$

6. TIME PERTURBATION - RECTANGULAR MONOPULS

In this case the “temporal perturbation - monopuls“ has $u(t)\equiv 1$, $\varphi_t:=Arccos(1)\equiv 0$, $u(t):\equiv\cos(0)=1$, $0=\theta_u:=\varphi_{t_0+T}-\varphi_{t_0}\equiv\omega_0\cdot T$ and the width of the time window of the perturbation is timed with a own **clock, locked** $\omega_0:=\theta_u/T=0$; $\theta(\omega_0,T)\equiv\theta_{\omega_0,T}\equiv\theta_u=0$ & $\theta/\theta_u=\omega/\omega_0=\infty$. In the relation (1) we have $\varphi'=0$ and by direct calculation (including the prolongation by continuity) we obtain $\underline{u}_{t_0,t_*}(\omega)=T\cdot e^{-i\cdot\left(\omega\cdot t_0+\frac{\theta}{2}\right)}\cdot\mathrm{sinc}\left(\frac{\theta}{2}\right)$; so

$$P^{m|p}_{\eta,t_*|t_0}\cong|\eta\cdot w^{p,m}|^2\cdot|\underline{u}_{t_0,t_*}(\omega^{p,m})|^2\cong|\eta\cdot w^{p,m}|^2\cdot(T/2)^2\cdot\mu^{\Pi}_T(\omega^{p,m})\equiv P^{m|p}_{\eta,T}$$

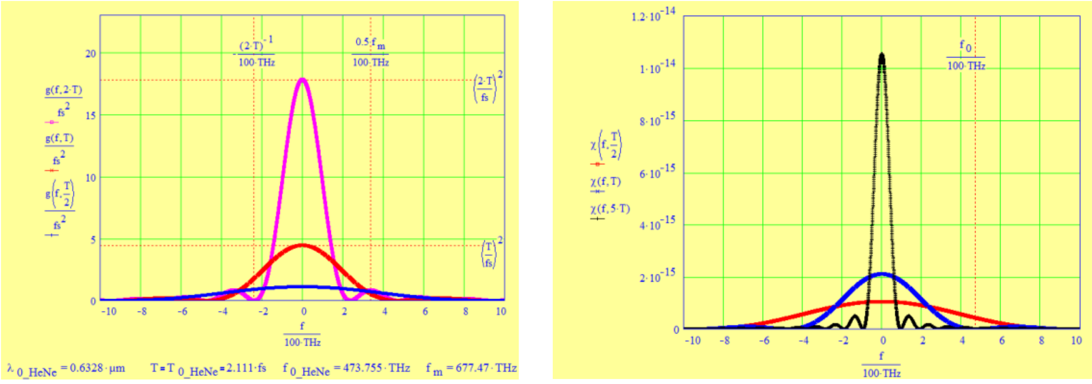
$$\mu^{\Pi}(\theta)=4\cdot\left[\mathrm{sinc}\left(\frac{\theta}{2}\right)\right]^2,\ \mu^{\Pi}_T(\omega):=4\cdot\left[\mathrm{sinc}\left(\omega\cdot\frac{T}{2}\right)\right]^2,\ \mu_{\Pi}(f,T):=4\cdot[\mathrm{sinc}(\pi\cdot f\cdot T)]^2$$

- The transition probability, with some versions, is:

$$P^{m|p}_{\eta,T}\cong|\eta\cdot w^{p,m}|^2\cdot g_T(\omega^{p,m}),\ g_T(\omega):=\left[T\cdot\mathrm{sinc}\left(\omega\cdot\frac{T}{2}\right)\right]^2,\ g(f,T)\equiv T\cdot\chi(f,T)$$

$$\chi(f,T):=T\cdot[\mathrm{sinc}(\pi\cdot f\cdot T)]^2\ \text{where}\ \int_{-\infty}^{+\infty}\chi(f,T)\cdot df\equiv 1\ \&\ \lim_{T\to\infty}\chi(f,T)=\delta(f)$$

The approximation with δ -Dirac distribution is often used in qualitative analyzes. We have MathCAD representations (fig. 1) for the laser THz domain. For a given frequency the transition probability varies periodically with the duration of perturbation (fig. 2).



(a) (b)

FIG. 1. Rectangular perturbation mono-pulse case: (a) transition probability factor in our laser frequencies domain frequency for different durations of perturbation; (b) δ -*Dirac* approximation

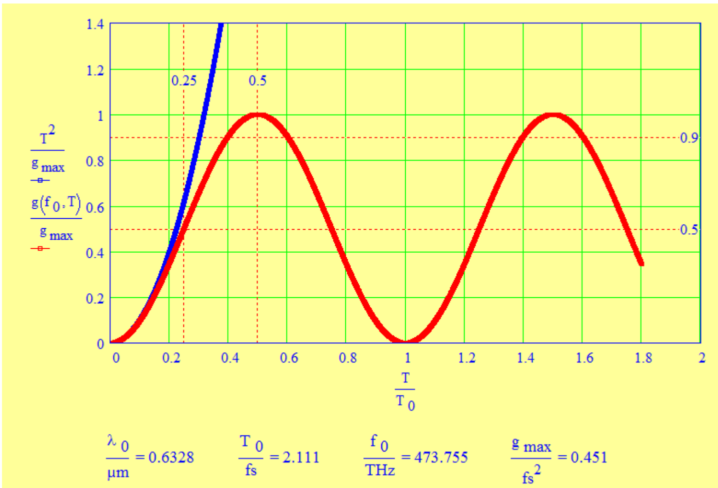


FIG. 2. Rectangular perturbation mono-pulse case:
transition probability factor (for a given laser frequency) vs duration duration of perturbation

7. TIME PERTURBATION – SINUSOIDAL MONOPULS

In this case we have $u(t) := \cos(\varphi_t)$, $\varphi_t := \omega_0 \cdot (t - t_0) + \varphi_0$; $\omega_0 = 2 \cdot \pi \cdot f_0$, $T_0 := 1/f_0$, $\theta_u := [\omega_0 \cdot (t_* - t_0) + \varphi_0] - \varphi_0 = \omega_0 \cdot T$, $\theta := \omega \cdot T$ and the width of the time window of the perturbation is timed with a own **clock** $\omega_0 := \theta_u / T$ & $\theta / \theta_u = \omega / \omega_0$. In the relation (1) $\varphi'_t = \omega_0$ and by direct calculation (including the prolongation by continuity) we obtain:

$$\underline{u}_{t_0,t_*}(\omega) = \frac{T}{2} \cdot \frac{e^{-i \cdot \omega_0 \cdot t_0}}{i} \cdot \begin{cases} \dfrac{e^{i \cdot (\theta_u - \theta)} - 1}{(\theta_u - \theta) \cdot e^{-i \cdot \varphi_{t_0}}} - \dfrac{e^{-i \cdot (\theta_u + \theta)} - 1}{(\theta_u + \theta) \cdot e^{i \cdot \varphi_{t_0}}} & |\omega| \neq \omega_0 \\ \dfrac{i}{e^{-i \cdot \varphi_0}} - \dfrac{e^{-i \cdot 2 \cdot \theta_u} - 1}{2 \cdot \theta_u \cdot e^{i \cdot \varphi_0}} & \omega = \omega_0 \\ \dfrac{e^{i \cdot 2 \cdot \theta_u} - 1}{2 \cdot \theta_u \cdot e^{-i \cdot \varphi_0}} + \dfrac{i}{e^{i \cdot \varphi_0}} & \omega = -\omega_0 \end{cases} , \quad \begin{matrix} \theta := \omega \cdot T \\ \theta_u := \omega_0 \cdot T \end{matrix}$$

- The transition probability, with some universal functions, is:

$$P^{m|p}_{\eta,t|t_0} \cong |\eta \cdot w^{p,m}|^2 \cdot |\underline{u}_{t_0,t_*}(\omega^{p,m})|^2 \equiv |\eta \cdot w^{p,m}|^2 \cdot (T/2)^2 \cdot \mu^S_T(\omega^{p,m}, \omega_0, \varphi_0) \equiv P^{m|p}_{\eta,T}$$

where, in complex format we have

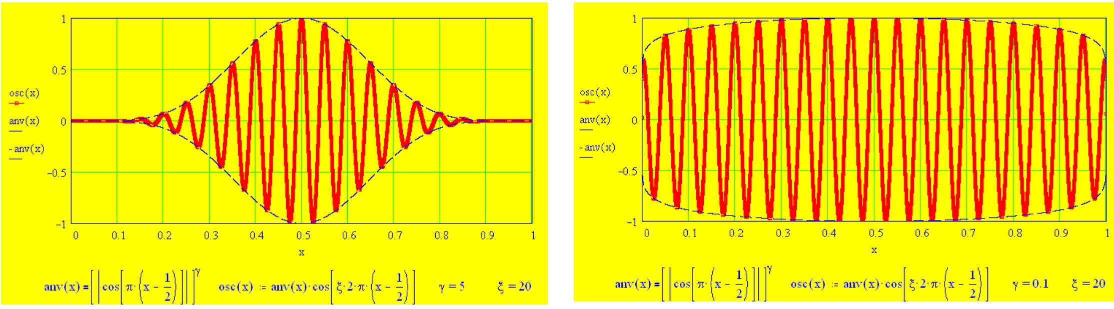
$$\mu_T^S(\omega,\omega_0,\varphi_0) := \mu^S(\omega\cdot T,\omega_0\cdot T,\varphi_0)\,, \qquad \mu^S(\theta,\theta_u,\varphi_0) := \left|\frac{u_{t_0,t_*}(\omega)}{T/2}\right|^2$$

$$\mu_S(f,T,f_0,\varphi_0) := \begin{cases} \dfrac{1}{4\cdot\pi^2\cdot T^2}\cdot\left|\dfrac{e^{i\cdot 2\pi\cdot(f_0-f)\cdot T}-1}{(f_0-f)\cdot e^{-i\cdot\varphi_0}} - \dfrac{e^{-i\cdot 2\pi\cdot(f_0+f)\cdot T}-1}{(f_0+f)\cdot e^{i\cdot\varphi_0}}\right|^2 & \mid f \mid \neq f_0 \\ \left|\dfrac{e^{i\cdot 4\cdot\pi\cdot f_0\cdot T}-1}{4\cdot\pi\cdot f_0\cdot T\cdot e^{-i\cdot\varphi_0}} + \dfrac{i}{e^{i\cdot\varphi_0}}\right|^2 & \mid f \mid = f_0 \end{cases}$$

or in real format

$$\mu_S(f,T,f_0,\varphi_{t_0}) := [\mathrm{sinc}(\pi\cdot(f_0-f))]^2 + [\mathrm{sinc}(\pi\cdot(f_0+f))]^2$$
$$+2\cdot\mathrm{sinc}(\pi\cdot(f_0-f))\cdot\mathrm{sinc}(\pi\cdot(f_0+f))\cdot\cos(2\cdot(\pi\cdot f_0\cdot T+\varphi_0))$$

This sinusoidal monopuls is an ideal approximation of a more realistic case (MathCAD represented in fig. 3). With reduced notation $\mu_S(f,T,f_0,\pi/2)\equiv\mu(f,T)$ we have (fig. 4) the frequency spectrum în the approximate idealized rectangular envelope for sinusoidal perturbation.



(a) (b)

FIG. 3. Sinusoidal perturbation mono-puls case:
(a) realistic sinusoidal envelope; (b) idealized rectangular envelope

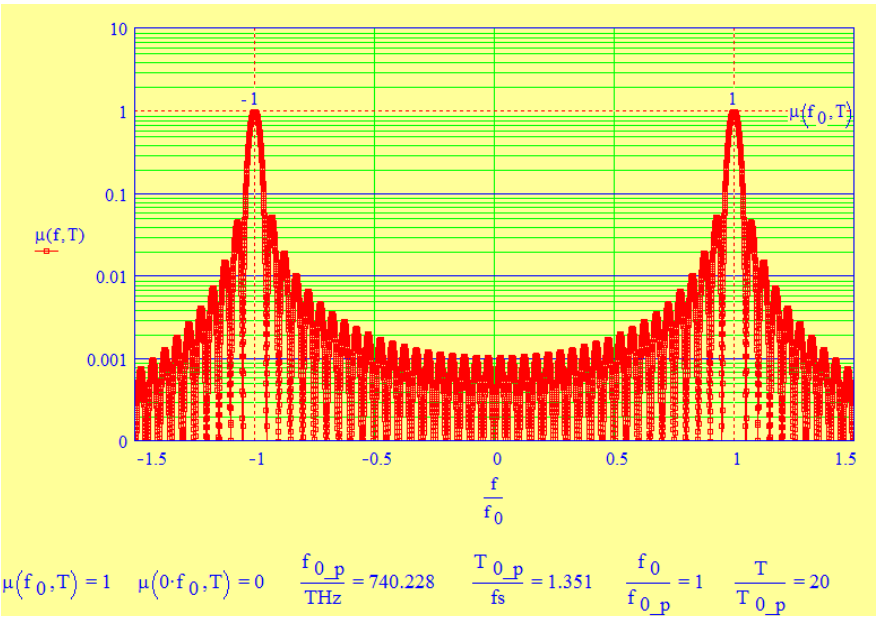


FIG. 4. The frequency spectrum
sinusoidal perturbation mono-puls case, idealized rectangular envelope

8. ON THT SECOND-ORDER APPROXIMATION

If the initial $\mathbf{s}_p$ & final $\mathbf{s}_m$ states ($p \neq m$) are <u>*not directly coupled*</u> by perturbation Hamiltonian $\mathbf{H}_{\zeta_{t,\eta}|a_t} \cong \eta \cdot \boldsymbol{\mathcal{H}}_{\zeta_{t,0}|a_t}$ (*1^st^ - order approximation* !) because $\mathcal{H}^{p,m}(t) \equiv 0$, but if $\mathbf{s}_p$ & $\mathbf{s}_m$ are <u>*indirectly coupled*</u> via a states $\mathbf{s}_n$ we evaluate (*2^nd^ - order approximation* !) $(\mathbf{s}_p \mapsto \mathbf{s}_m) = \sum_n (\mathbf{s}_p \mapsto \mathbf{s}_n \mapsto \mathbf{s}_m)$ in the context of the superposition principle of quantum physics states, as follows:

$P^{m|p}_{\eta,t|t_0} = | c^{m|p}_{a_t \oplus \zeta_{t,\eta}} |^2 = | \overline{c^{m|p}_{a_t \oplus \zeta_{t,\eta}}} |^2$, $c^{m|p}_{a_t \oplus \zeta_{t,\eta}} = \gamma^{m|p}_{a_t \oplus \zeta_{t,\eta}} \cdot e^{-i \cdot \omega_m \cdot (t - t_0)}$, $\gamma^{m|p}_{a_t \oplus \zeta_{t,\eta}} \cong \eta^2 \cdot \varsigma^{m|p}_{t,2}$ and

$i \cdot \hbar \cdot \dfrac{d\varsigma^{m|p}_{t,2}}{dt} = \sum_{n \neq m} [e^{-i \cdot \omega^{n,m} \cdot t_0} \cdot \overline{\mathcal{H}^{n,m}(t)} \cdot e^{i \cdot \omega^{n,m} \cdot t} \cdot \varsigma^{n|p}_{t,1}]$, *2^nd^ - order approximation*, based on:

$\varsigma^{n|p}_{t,1} = \dfrac{e^{-i \cdot \omega^{p,n} \cdot t_0}}{i \cdot \hbar} \cdot \int_{t_0}^{t} \overline{\mathcal{H}^{p,n}(t')} \cdot e^{i \cdot \omega^{p,n} \cdot t'} \cdot dt' \neq 0$ *1^st^ - order approximation;* by integration:

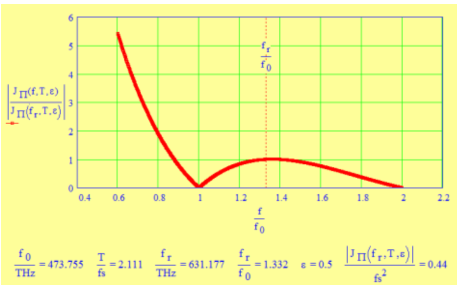
$$\varsigma^{m|p}_{t,2} = \frac{e^{-i \cdot \omega^{p,m} \cdot t_0}}{(i \cdot \hbar)^2} \cdot \sum_n \left[\int_{t_0}^{t} dt'' \cdot \int_{t_0}^{t''} dt' \cdot \overline{\mathcal{H}^{p,n}(t') \cdot \mathcal{H}^{n,m}(t'')} \cdot e^{i \cdot (\omega^{p,n} \cdot t' + \omega^{n,m} \cdot t'')} \right]$$

$$P^{m|p}_{\eta,t_*|t_0} \cong \left(\frac{\eta}{\hbar} \right)^4 \cdot \left| \sum_{p \neq n \neq m} \left[\int_{t_0}^{t_*} dt'' \cdot \int_{t_0}^{t''} dt' \cdot \mathcal{H}^{p,n}(t') \cdot \mathcal{H}^{n,m}(t'') \cdot e^{-i \cdot (\omega^{p,n} \cdot t' + \omega^{n,m} \cdot t'')} \right] \right|^2$$

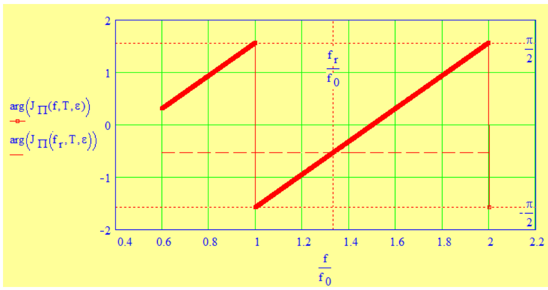
The approximate calculation of this iterated integration has led (for example in the rectangular monopuls, single intermediate energy level at $\varepsilon = 50 \cdot \%$) to the relationship:

$P^{m|p}_{\eta,t_*|t_0} \cong \eta^4 \cdot | w^{p,n} \cdot w^{n,m} |^2 \cdot |J_\Pi|^2$, $J_\Pi(f, T, \varepsilon) := \dfrac{2 \cdot \sin(\pi \cdot f \cdot T)}{i \cdot \varepsilon^2 \cdot (2 \cdot \pi \cdot f)^2} \cdot e^{i \cdot \pi \cdot f \cdot T}$

The maximization of the transition does not coincide with the laser frequency (fig. 5).



(a)



(b)

FIG. 5. Second order approximation factor (of transition probability) frequency analysis rectangular perturbation mono-pulse case: (a) normalized modulus; (b) phase of factor

9. CONCLUSIONS

- Both mathematical modeling and numerical simulation of stimulated transition probabilities for quantum optics have been performed.
- Revised perturbation theory equations in the states spectrum of a quantum system have been established in order to evaluate the stimulated transition probability (in the quantum physics Hilbert space).

- The formalization is in accordance with the formal framework of information theory (regarding: entropy, conditional entropy and mutual information adapted to the Hamiltonian Formalism).
- The operatorial relationships have been used distinctly from matrix-type relationships (Dirac formalism with “bra” and “ket”) and have been intended exclusively for the numerical simulation.
- Analytical relations have been rewritten and systematized
- Particular temporal patterns, (quasi)-rectangular or (envelope)-sinusoidal, mono-pulse of the perturbation and corresponding transition probabilities were analyzed and represented normalized by MathCAD.

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