BERNSTEIN POLYNOMIALS IN THE APPROXIMATION AND CONVERGENCE OF DERIVABLE FUNCTIONS

Florența Violeta TRIPŞA

”*Transilvania*” University, Braşov, Romania (florentatripsa@yahoo.com)

DOI: 10.19062/2247-3173.2018.20.40

Abstract *This article points out the properties that the Bernstein polynomials have, related to the approximation of the derivable functions and their convergence, together with the elegant results obtained through problem solving. A continuous function with its first k derivatives is considered on the interval* (a,b) *and it is proven that the Bernstein polynomial arrays* $B_n(f;x)$, $B'_n(f;x)$,..., $B_n^{(k)}(f;x)$ *tend absolutely and uniformly towards the functions* $f(x)$, $f'(x)$,..., $f^{(k)}(x)$ *respectively, on the entire* (a,b) *interval.*

Keywords: *approximation, polynomials, functions, continuity, boundedness*

1. INTRODUCTION

Let $f(x)$ be a continuous function on the (a,b). We divide the interval (a,b) in *n* equal parts and we get:

$$x_i = a + i\frac{b-a}{n} \tag{1}$$

Division points, where $i = 0,1,\ldots,n$, with $x_0 = a$; $x_n = b$.

An *n* degree polynomial whose coefficients depend linearly and homogenously on the *(n+1)* values $f(x_i)$ with $i = \overline{0,n}$ is called an *n* degree interpolation polynomial of the $f(x)$ function.

We note with

$$B_n(f;x) = \frac{1}{(b-a)^n}\sum_{i=0}^{n} C_n^i f(x_i)\cdot(x-a)^i(b-x)^{n-i} \tag{2}$$

The Bernstein interpolation polynomial.

Also, the oscillation module of the *f* function is defined by

$$\omega(\delta) = \max\left|f(x') - f(x'')\right| \tag{3}$$

Where x' and x'' are two ordinary points of the (a,b) interval with the property

$$\left|x' - x''\right| \le \delta \text{, where } \quad \delta \in (0, b-a] \tag{4}$$

We mark the divided difference by the order $(k-1)$ of the *f* function in the points $x_1, x_2, \ldots, x_k$ with

$$D_k = \frac{U(x_1, x_2, \ldots, x_k; f)}{V(x_1, x_2, \ldots, x_k)} \tag{5}$$

As being the relation between the two determinants: $U(x_1, x_2, ..., x_k; f)$ being the determinant obtained from the Vandermonde determinant of $V(x_1, x_2, ..., x_k)$ of the numbers $x_1, x_2, ..., x_k$, replacing the last respective column with the elements $f(x_1), f(x_2), ..., f(x_k)$ and the determinant $V(x_1, x_2, ..., x_k)$.

We note $D_k = [x_1, x_2, ..., x_k; f]$ the difference divided by the $(k-1)$ order of the $f(x)$ function.

We note with $D_n[f] = \max\limits_{(a,b)} |[x_1, x_2, ..., x_{n+1}; f]|$ the limit of the n order of the f function in (a,b) interval, where $x_1, x_2, ..., x_{n+1}$ are $(n+1)$ random distinct points of the (a,b) interval.

If f admits a bounded derivative of an $(n+1)$ order and if we note with $D_0[f^{(n+1)}]$ the maximum or the superior limit of $\left|f^{(n+1)}\right|$ in the interval (a,b) we have

$$D_0[f^{(n+1)}] = (n+1)! D_{n+1}[f] \tag{6}$$

2. THE APPROXIMATION OF THE DERIVATIVE FUNCTIONS

Let's assume that the f function has a continuous derivative of k order and let $\omega_k(\delta)$ be the oscillation module of this derivative.

It is known that we have the generalized mean formula:

$$k! D_k^i = f^{(k)}\left(a + \frac{b-a}{n}(i + \theta k)\right), 0 < \theta < 1 \tag{7}$$

where $D_k^i = [x_i, x_{i+1}, ..., x_{i+k}; f]$, $i = 0,1,...,n-k$, $k = 1,2,...$

$$\left|k! D_k^i - f^{(k)}\right| \le \omega_k\left(\left|x - a - \frac{b-a}{n}(i + \theta k)\right|\right) \le \omega_k\left(\max\left(|x - x_i|, |x - x_{i+1}|, ..., |x - x_{i+k}|\right)\right) \tag{8}$$

In the following, we consider the polynomial:

$$Q_{n,k}(f;x) = \frac{B_n^{(k)}(f;x)}{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)...\left(1 - \frac{k-1}{n}\right)} = \frac{1}{(b-a)^{n-k}} \sum_{i=0}^{n-k} C_{n-k}^i k! D_k^i (x-a)^i (b-x)^{n-k-i} \tag{9}$$

We have

$$\left|f^{(k)} - Q_{n,k}(f;x)\right| < \left\{\frac{1}{\delta}\left(\frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{s} C_{n-k}^i |x - x_i| (x-a)^i (b-x)^{n-k-i} + \right.\right.$$
$$\left.\left. + \frac{1}{(b-a)^{n-k}} \cdot \sum_{i=s+1}^{n-k} C_{n-k}^i |x - x_{i+k}| (x-a)^i (b-x)^{n-k-i}\right) + 1\right\} \cdot \omega_k(\delta) \tag{10}$$

Where s is determined as follows:

$$s = j - \frac{k}{2} \text{ if } k \text{ is even and } x_j \le x \le x_{j+1} \tag{11}$$

$$s = j - \frac{k+1}{2} \text{ If } k \text{ is odd and } x_j \le x \le \frac{x_j + x_{j+1}}{2} \tag{12}$$

$$s = j - \frac{k-1}{2} \text{ if } k \text{ is odd and } \frac{x_j + x_{j+1}}{2} \le x \le x_{j+1} \tag{13}$$

If in these formulae we have $s < 0$ or $s \geq n-k$, either the first or the second member in the second parenthesis of the (10) relation disappears.

We notice that

$$|x - x_{i+k}| \leq |x - x_i| + |x_{i+k} - x_i| - |x - x_i| + \frac{k(b-a)}{n} \tag{14}$$

$$\left|f^{(k)} - Q_{n,k}(f;x)\right| = \left\{\frac{1}{\delta}\left(\frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}|x - x_i|(x-a)^i(b-x)^{n-k-i} + \Psi(x)\right) + 1\right\} \cdot \omega_k(\delta) \tag{15}$$

Where

$$\Psi(x) = \frac{k}{n} \cdot \frac{1}{(b-a)^{n-k-1}} \cdot \sum_{i=s+1}^{n-k} C_{n-k}^{i}(x-a)^i(b-x)^{n-k-i} \tag{16}$$

And we must consider $\Psi(x) = 0$ if $s \geq n-k$.

Let's further notice that

$$\begin{aligned}\frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}|x - x_i|(x-a)^i(b-x)^{n-k-i} \leq \\ \leq \frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}\left|x - a - i\frac{b-a}{n-k}\right|(x-a)^i(b-x)^{n-k-i} + \\ + \frac{k}{n(n-k)(b-a)^{n-k-1}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i} i (x-a)^i(b-x)^{n-k-i}\end{aligned} \tag{17}$$

That results from the relation

$$x - x_i = x - a - i\frac{(b-a)}{n} + i\frac{k(b-a)}{n(n-k)} \tag{18}$$

It is known that

$$\frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}\left|x - a - i\frac{b-a}{n-k}\right|(x-a)^i(b-x)^{n-k-i} \leq \frac{b-a}{2\sqrt{n-k}} \tag{19}$$

On the other hand

$$\frac{k}{n(n-k)(b-a)^{n-k-1}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i} i (x-a)^i(b-x)^{n-k-i} = \frac{k(x-a)}{n} \tag{20}$$

In these conditions we write:

$$\frac{1}{(b-a)^{n-k}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}|x - a_i|(x-a)^i(b-x)^{n-k-i} \leq \frac{b-a}{2\sqrt{n-k}} + \frac{k(b-a)}{n} \tag{21}$$

We obviously obtain for the function $\Psi(x)$ the inequality:

$$\Psi(x) \leq \frac{k}{n} \cdot \frac{1}{(b-a)^{n-k-1}} \cdot \sum_{i=0}^{n-k} C_{n-k}^{i}(x-a)^i(b-x)^{n-k-i} = \frac{k(b-a)}{n} \tag{22}$$

Thus resulting in:

$$\left|f^{(k)} - Q_{n,k}(f;x)\right| < \left\{\frac{1}{\delta}\left(\frac{b-a}{2\sqrt{n-k}} + \frac{2k(b-a)}{n}\right) + 1\right\} \cdot \omega_k(\delta) \tag{23}$$

We consider $\delta = \dfrac{b-a}{\sqrt{n-k}}$ obtaining:

$$\left|f^{(k)} - Q_{n,k}(f;x)\right| < \left(\frac{3}{2} + \frac{2k\sqrt{n-k}}{n}\right)\omega_k\left(\frac{b-a}{\sqrt{n-k}}\right) \leq \frac{3+2\sqrt{k}}{2}\,\omega_k\left(\frac{b-a}{\sqrt{n-k}}\right) \tag{24}$$

with $n \geq k+1$.

3. THE CONVERGENCE OF BERNSTEIN POLYNOMIALS DERIVATIVES

If $D_k^i = [x_i, x_{i+1}, \ldots, x_{i+k}; f]$, $i = 0,1,\ldots,n-k$, $k = 1,2,\ldots$ a simple calculus shows that

$$B_n^{(k)}(f;x) = \frac{dB_n(f;x)}{dx} = \frac{1}{(b-a)^{n-1}} \sum_{i=0}^{n-1} C_{n-1}^i D_1^i (x-a)^i (b-x)^{n-1-i} \tag{25}$$

And generally:

$$\frac{d^k B_n(f;x)}{dx} = k!\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\frac{1}{(b-a)^{n-k}} \sum_{i=0}^{n-1} C_{n-k}^i D_k^i (x-a)^i (b-x)^{n-k-i} \tag{26}$$

As the derivative of a k order of function f is supposed to be continuous, the superior border $D_0[f^{(k)}]$ is then finite.

We write

$$f^{(k)} - B_n^{(k)}(f;x) = f^{(k)} - Q_{n,k}(f;x) + \left[1-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\right]\cdot Q_{n,k}(f;x) \tag{27}$$

How

$$\left|Q_{n,k}(f;x)\right| \le D_0\left[f^{(k)}\right] \tag{28}$$

And

$$1-\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right) \tag{29}$$

Taking into consideration the relation (24), we deduce:

$$\left|f^{(k)} - B_n^{(k)}(f;x)\right| < \frac{3+2\sqrt{k}}{2}\cdot\omega_k\left(\frac{b-a}{\sqrt{n-k}}\right) + \frac{k(k-1)}{2n}\cdot D_0\left[f^{(k)}\right] \tag{30}$$

This shows that:

If function f, definite in the interval (a,b), is continuous together with its first k derivatives, the polynomial arrays $B_n(f;x)$, $B'_n(f;x), \ldots,$ $B_n^{(k)}(f;x)$ tend absolutely and uniformly towards $f(x)$, $f'(x), \ldots,$ $f^{(k)}(x)$ respectively, on the entire (a,b) interval.

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