APPLICATION OF PROCEDURES FOR RELATIONS ESTABLISHMENT BETWEEN NATURAL NUMBERS

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Abstract: *By adding consecutive natural numbers was developed an algebraic structure with levels called Natural procedure.It has been shown that the powers of natural numbers can be expressed in terms of these values. Double recurrence requires the concept of procedure and defining it as increasing or decreasing structure. A particular case of procedure is the coefficients of the sum writing for natural numbers powers in Natural Procedure.Combinations to obtain zero in this system are analyzed. It exemplifies the types of procedures for achieved algebraic developments.*

Key words: *powers of natural numbers, procedure, obtaining null relations, algebraic developments.*

1. INTRODUCTION

Procedures are two variables natural or integers recurrent rows. These have theoretical significance and structure with levels. This paper will present and discuss results on the definition of such procedures.

The area of study is situated in the combinatorial analysis.This paper discussed and given a theoretical and practical sense for presented relations. This paper presents the main results of works [1-8]. To these followed a methodical approach showing the importance of each relation separately.

2. NATURAL PROCEDURE

Natural procedure was given by summing consecutive natural numbers. Results obtained were treated as "natural numbers" and apply the same type of addition. Furthermore it was shown that there are two previous levels of original level considered.

We obtained the following definition relationship:

$$s_{-1}(n) = 1 \ \ \forall n \in N^*$$

$$s_0(n) = 1 + 1 + \ldots + 1_{(n\, times)} = n$$

$$s_1(n) = 1 + 2 + 3 + \ldots + n$$

$$s_2(n) = s_1(1) + s_1(2) + s_1(3) + \ldots + s_1(n)$$

$$s_3(n) = s_2(1) + s_2(2) + s_2(3) + \ldots + s_2(n)$$

....

$$s_t(n) = s_{t-1}(1) + s_{t-1}(2) + s_{t-1}(3) + \ldots + s_{t-1}(n) \tag{1}$$

For realized sums was formulated the recurrence relation:

$$s_t(n) = s_t(n-1) + s_{t-1}(n) \tag{2}$$

It was discovered a formula as a product that allows calculation of the sums $s_t(n)$:

$$s_t(n) = \frac{n(n+1)(n+2)\cdot\ldots\cdot(n+t)}{(t+1)!} \tag{3}$$

Equation (3) satisfies the recurrence (2) even for integer values of n.Based on primary results was defined Upper Natural Procedure (defined above -1 level of t).

$$s_{-1}(n) = 1 \ \ \forall n \in Z$$

(the initial value of the variable)

$$s_t(1) = 1 \ \ \forall t \in Z\, t \geq -1$$

(set value for each level)

$s_t(n) = s_t(n-1) + s_{t-1}(n)$ or relation (3)

(recurence) (4)

The three relations define superior natural procedure.

The most important application of the natural procedure is the decomposition of powers of natural numbers.

3. UPPER NATURAL PROCEDURE AND RIGHT NATURAL PROCEDURE

The sums of upper natural procedure form a symmetrical array given by relation:

$$s_t(n) = s_{n-2}(t+2) \ \forall n,t \in N^* \qquad (5)$$

Natural procedure allows generalization to: $\forall n,t \in Z, t \geq -1$ This generalization defines Upper Natural Procedure described above. This is the given by recurrence and the relation:

$$s_{-1}(n) = 1 \ \forall n \in \mathrm{Z} \cdot \qquad (6)$$

It is considered that at each level there is only t+2 values of sums are defined. Thus at t=-1 there is only one no value defined and procedure recurrence is not applicable. In other words, the procedure requires that the value of n does not exceed t count of -1.

Level t = -2 is independent. Provided that t and n have the same increase does not have algebraic bases. Knowing this avoid contradictions in defining the level t = -2.

Right Natural procedure is an image of Upper Natural Procedure that reverses the n to t.It is given by:

$$s_t(1) = 1 \ \forall n,t \in \mathrm{Z} \ \ n \geq 1$$

$$s_t(n) = s_t(n-1) + s_{t-1}(n) \ \ \forall n,t \in \mathrm{Z} \ \ n \geq 1 \qquad (7)$$

For negative values of t it is given by:

$$s_{-t-1} = \frac{(-t+1)(-t+2)\cdot \ldots \cdot(-t+n-1)}{(n-1)}$$

$$\forall t \in N, n \in N^* \qquad (8)$$

Property 1: With condition of correlation increasing for n variable with t variable Right Natural Procedure and Upper Natural Procedure may be accepted simultaneously.

For $n \in Z$ we have:

$$s_{-1}(n) = s_{n-2}(1) = 1 \qquad (9)$$

Correlation between initially levels for that was applied recurence (3) is:

$$s_{-1}(0) = s_{-2}(1) = 1 \qquad (10)$$

$$s_t(0) = s_{-2}(t+2) = 0 \ \ t \geq 0 \qquad (11)$$

These relation define level t=-2, independent for positive n. It was not defined in the negative field. It is shown that if would be defined $s_{-2}(0)$ would be its own symmetric given by (5).

The name of two procedures comes from the location of their definition values to a coordinate system with axes n abscissa axis and t of ordinates axis.

4. POWERS OF NATURAL NUMBERS

Power t+1 of a natural number is written on the basis of consecutive sums of Natural Procedure at t level as follows:

$$n^{t+1} = \sum_{i=0}^{t} a_t^i s_t(n-i) \qquad (12)$$

Where the coefficients a_t^i are given by the following recurrence:

$$a_o^o = 1 \text{ şi } a_o^i = 0 \ \ \forall i \in Z \ i \neq 0$$

$$a_t^i = (i+1) \ a_{t-1}^i + (t+1-i)a_{t-1}^{i-1} \ \ \forall i \in Z \ \ 0 \leq i \leq t \qquad (13)$$

Coefficients a_t^i represent an increasing procedure.

Following relations presents some examples of decomposition of powers of natural numbers. With $\forall n \in N$ (can be extended to $\forall n \in Z$)

$$n^2 = s(n) + s(n-1)$$

$$n^3 = s_2(n) + 4s_2(n-1) + s_2(n-2)$$

$$n^4 = s_3(n) + 1 \ \ s_3(n-1) + 1 \ \ s_3(n-2) + s_3(n-3)$$

$$n^5 = s_4(n) + 26s_4(n-1) + 6 \ \ s_4 s(n-2) + 26s_4(n-3) + s_4(n-1)$$

$$n^6 = s_5(n) + 57s_5(n-1) + 302s_5(n-2) + 302s_5(n-3) + 57s_5(n-4) + s_5(n-5)$$

$$n^7 = s_6(n) + 120s_6(n-1) + 1191s_6(n-2) + 2416s_6(n-3) + 1191s_6(n-4) + 120s_6(n-5) + s_6(n-6) \qquad (14)$$

The procedure a_t^i is one increasing after t defined by its first level and recurrence.

For it was observed the following:

- it is symmetric;

$$a_t^i = a_t^{t-i} \ \ \forall i \in Z \ \ 0 \leq i \leq t \qquad (15)$$

- nonzero values beginning with 1;

$$a_t^0 = a_t^t = 1 \ \ \ \forall i \in Z \ \ 0 \leq i \leq t \qquad (16)$$

-procedures has at level t nonzero t+1 values

$a_t^i \neq 0 \;\; \forall i \in Z \;\; 0 \leq i \leq t$ şi (17)

$a_t^i = 0 \quad \forall i \in Z \quad \forall i \in Z \;\; i < 0 \quad \forall i \in Z$ o r $i. > t$ (18)

-sum of coefficients a_t^i is a factorial number:

$$\sum_{i=0}^{t} a_t^i = (t+1)! \;\; \forall t \in Z, t \geq 0 \qquad (19)$$

An important consequence of the writings of powers of natural numbers is the easy calculation of the sum of consecutive natural numbers powers:

$$S_{t+1}(n) = 1^{t+1} + 2^{t+1} + ... + n^{t+1} = \sum_{i=0}^{t} a_t^i s_{t+1}(n-i) \qquad (20)$$

Writing in sum for powers of natural numbers (12) can be compared with that known given by the recurrence:

$$(n+1)^{t+1} = 1 + C_{t+1}^1 S_t(n) + C_{t+1}^2 S_{t-1}(n) + C_{t+1}^3 + ... + C_{t+1}^t S_1(n) + n$$

(21)

This can be written as a sum, in a manner similar to equation (12).

$$n^{t+1} = 1 + \sum_{k=0}^{t} C_{t+1}^k S_k(n-1) \qquad (22)$$

5. FORMULAS FOR MULTIPLICATION AND DIFFERENCES

The main observation is that relations for writing natural numbers powers (12) is independent of quality natural number for n and the it is applicable to real or complex numbers $n \to x$. Coefficients a_t^i are described only by the variable t. Acting on it condition that it is natural.

There have been deducted a number of important relations. Multiplication is given by the the following relation:

$$a \cdot b = s(a+b) - s(a) - s(b) \;\; \forall a, b \qquad (23)$$

You can define generalized multiplication formula:

$$\sum_{k=0}^{t-1} s_k(a) \cdot s_{t-1-k}(b) = s_t(a+b) - s_t(a) - s_t(b) \;\; \forall a, b \qquad (24)$$

The difference formula is based on the relation:

$$n^{t+1} - (n-1)^{t+1} = \sum_{i=0}^{t+1} a_t^i s_{t-1}(n-i) \qquad (25)$$

This keeps coefficients a_t^i but decreases the the index sum attached.This decrease of the sum can be generalized by introducing *differences of differences*. This is following relation:

$$^{k+1}\Delta_n^{t+1} \overset{def}{=} {}^{k}\Delta_n^{t+1} - {}^{k}\Delta_{n-1}^{t+1} \qquad (26)$$

With:

$$^{0}\Delta_n^{t+1} \overset{def}{=} n^{t+1} \qquad (27)$$

The differences can be easily calculated according to n by following relation:

$$^{k}\Delta_n^{t+1} = \sum_{j=0}^{k} C_k^j (-1)^j (n-j)^{t+1} \qquad (28)$$

The differences are expressions with power $t+1$ of consecutive natural numbers written backwards from n.

Applying writing powers of natural numbers in the system coefficients a_t^i we obtain the relation:

$$\sum_{j=0}^{k} C_k^j (-1)^j (n-j)^{t+1} = \sum_{i=0}^{t} a_t^i s_{t-k}(n-i) \qquad (29)$$

Order difference k can grow indefinitely. Thus we have two particular applications of this relation:

To $k = t+1$ have:

$$\sum_{j=0}^{k} C_{t+1}^j (-1)^j (n-j)^{t+1} = \sum_{i=0}^{t} a_t^i = (t+1) \qquad (30)$$

To $k \geq t+2$ have:

$$\sum_{j=0}^{k} C_k^j (-1)^j (n-j)^{t+1} = 0 \qquad (31)$$

This show that there is a sum of t+1 power for over t+1 consecutive integers form a preset null combination.

6. THE PRODUCT OF CONSECUTIVE INTEGER NUMBERS

Into decomposition of powers of natural numbers appears term $s_t(n-i)$. For this we have a write product and one in sum.

$$s_t(n-i) = \frac{(n-i)(n-i+1)(n-i+2)...(n-i+t)}{(t+1)!}$$
(32)

$$s_t(n-i) = \frac{1}{(t+1)!}\prod_{k=0}^{t}(n-i+k) \quad (33)$$

$$s_t(n-i) = \frac{1}{(t+1)!}\sum_{j=0}^{t+1} R_{i,t}^{j} n^{t+1-j} \quad (34)$$

The calculation of polynomial coefficients $R_{i,t}^{j}$ show a recurrent development.This calculation is based on considering the set:

$$M_t^i = \{-i, -i+1, -i+2, ..., -i+t\} \quad (35)$$

The elements of this set we note general with m_t^i. Coefficients $R_{i,t}^{j}$ will be calculated as the sum of all products of size j.for elements form M_t^i.A general writing of this relationship is:

$$R_{i,t}^{j} = \sum_{C_t^j \; terms} \prod_{j \; terms} m_t^i \quad (36)$$

The following are some applications of calculating these coefficients.

$$(n-i)(n-i+1) = n^2 + [(-i)+(-i+1)]\, n + (-i)(-i+1) \quad (37)$$

$$\begin{aligned}&(n-i)(n-i+1)(n-i+2)\\ &= n^3 + [(-i)+(-i+1)+(-i+2)]n^2 + \\ &[(-i)(-i+1)+(-i)(-i+2)+(-i+1)(-i+2)]n\\ &+(-i)(-i+1)(-i+2)\end{aligned} \quad (38)$$

Coefficients $R_{i,t}^{j}$ can be written as a polynomial of the variable i. Thus from previous developments can identify coefficients $R_{i,t}^{j}$:

$t=1$ $R_{i,1}^{0} = 1$, $R_{i,1}^{1} = -2i+1$, $R_{i,1}^{1} = i^2 - i$ (39)

$t=2$ $R_{i,2}^{0} = 1$, $R_{i,2}^{1} = -3i+3$ (40)

$R_{i,2}^{2} = 3i^2 - 6i + 2$, $R_{i,2}^{3} = -i^3 + 3i^2 - 2i$ (41)

For developments follows that have by definition:

$$R_{i,t}^{0} = 1 \quad (42)$$

It is noted that relation takes place:

$$M_t^{i+1} \cap M_t^i = M_{t-1}^i \quad (43)$$

Based on this relationship is deduced following recurrence formula for coefficients $R_{i,t}^{j}$:

$$R_{i,t}^{j} - R_{i+1,t} = (t+1)\, R_{i,t-1}^{j} \quad (44)$$

This relation is important because it allows working with coefficients $R_{i,t}^{j}$ other than definition or direct calculation. It is an example of decreasing procedure (after t).

The values at the higher level generates the values on the lower level.It has been shown that takes place the relation:

$$\sum_{i=0}^{t} a_t^i R_{i,t}^{j} = 0 \;\; \forall\, 0 < j \le t \quad (45)$$

This represents t relations for annulment in coefficients system a_t^i. For this relation will present checks:

t=1 j=1 $\sum_{i=0}^{1} a_1^i R_{i,1}^{1} = 1\cdot 1 + 1\cdot(-1) = 0$ (46)

t=1 j=2 $\sum_{i=0}^{1} a_1^i R_{i,1}^{2} = 1\cdot 0 + 1\cdot 0 = 0$ (47)

It is noted that the relation for t, j=t+1 is identical zero and can not be used to characterize the coefficients a_t^i.

For t=2 j=1 we have:

$$\sum_{i=0}^{2} a_1^i R_{i,2}^{1} = 1\cdot 3 + 4\cdot 0 + 1\cdot(-3) = 0 \quad (48)$$

For t=2 j=1 we have:

$$\sum_{i=0}^{2} a_1^i R_{i,2}^{2} = 1\cdot 2 + 4\cdot(-1) + 1\cdot 2 = 0 \quad (49)$$

In the following we verify the functionality of the formulas presented in the calculation of n^{t+1}

$$n^{t+1} = \sum_{i=0}^{t} a_t^i s_t(n-i) =$$

$$\sum_{i=0}^{t} a_t^i \frac{1}{(t+1)!}\sum_{j=0}^{t+1} R_{i,t}^{j} n^{t+1-j} =$$

$$\frac{1}{(t+1)!}\sum_{i=0}^{t}\sum_{j=0}^{t+1} R_{i,t}^{j}\, a_t^i n^{t+1-j} =$$

$$\frac{1}{(t+1)!}\sum_{j=0}^{t+1}\sum_{i=0}^{t} R_{i,t}^{j}\, a_t^i n^{t+1-j} =$$

$$\frac{1}{(t+1)!}\sum_{j=0}^{t+1}\sum_{i=0}^{t} R_{i,t}^{j}\, a_t^i n^{t+1-j} =$$

$$\frac{1}{(t+1)!}\sum_{j=0}^{t+1} n^{t+1-j}\sum_{i=0}^{t} R_{i,t}^{j}\, a_t^i =$$
$$\frac{1}{(t+1)!} n^{t+1}\sum_{i=0}^{t} R_{i,t}^{0} a_t^i +$$
$$\frac{1}{(t+1)!}\sum_{j=1}^{t+1} n^{t+1-j}\sum_{i=0}^{t} R_{i,t}^{j}\, a_t^i =$$
$$\frac{1}{(t+1)!} n^{t+1}\sum_{i=0}^{t} 1\cdot a_t^i + \frac{1}{(t+1)!}\sum_{j=1}^{t+1} n^{t+1-j}\cdot 0 =$$
$$\frac{1}{(t+1)!} n^{t+1}\cdot (t+1)! = n^{t+1} \tag{50}$$

7. DETERMINATION OF a_t^i COEFFICIENTS BY THE $R_{i,t}^j$ COEFFICIENTS

Consider the system:

$$\sum_{i=0}^{t} a_t^i R_{i,t}^j = 0 \quad 0 \le i \le t \quad 1 \le j \le t \tag{51}$$

(t relations)

$$\sum_{i=0}^{t} a_t^i = (t+1)! \quad j,i,t \in N \tag{52}$$

(relation t+1 for j=0)

Property 2: Coefficients a_t^i are obtained as unique solution of the system of t+1 relations with t+1 unknowns, the system is compatible determined.

It is shown that the determinant of the system matrix verifies the following recurrence:

$$\Delta_t = (-1)^{t-1}(t+1)^t\,\Delta_{t-1} \;\; \forall t \in N^* \tag{53}$$

and by direct calculation we have first term of the recurrence:

$$\Delta_1 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \ne 0 \tag{54}$$

Thus for any natural number t coefficients a_t^i are uniquely determined in the coefficients system $R_{i,t}^j$.

Assume the presence of a row of numbers with the property that:

$$\sum_{i=0}^{t} a_t^i A_t^i = 0 \tag{55}$$

Substituting this relation t+1 with relation we obtain a homogeneous system that to have t solutions other of the trivial solution and no coefficients a_t^i exist, we have:

$$A_t^i = \sum_{j=1}^{t} w_t^j R_{i,t}^j \tag{56}$$

In other words, some values A_t^i are linear combinations of coefficients $R_{i,t}^j$. Algebraic relations not say if the w_t^j coefficients are integers or irrational (where A_t^i are integers).

8. DECREASING PROCEDURE α_t^i

Consider a fixed level and the sum

$$\sum_{i=0}^{t} a_t^i s_t(n-i).$$

Recurrence natural procedure allows the calculation of a lower level sums as differences of consecutive sums after i at the upper level.

This downward trend has led to the idea to define a set of coefficients α_t^i at t level and then by downwards procedure to get access at lower levels .

We have:

1. The sum $\sum_{i=0}^{t} a_t^i \alpha_t^i$ shows itself utility

2.Define the coefficients α_t^i for $0 \le i \le t, i \in N$ with definition property for the lower level:

$$\alpha_t^i - \alpha_t^{i+1} = \alpha_{t-1}^i \tag{57}$$

This has the generalization:

$$\alpha_p^i - \alpha_p^{i+1} = \alpha_{p-1}^i \;\; 0 \le p < t,\; p \in N \tag{58}$$

Property 3: Definition of coefficients α_t^i occurs for $0 \le i \le p, i \in N$. This definition is decreasing relative to the t level. (like triangle with pointing down).

Property 4: (lowering into the system of coefficients a_t^i)

In the system of coefficients a_t^i a relation type

$\sum_{i=0}^{t} a_t^i \alpha_t^i$ has the image at level a_{t-1}^i given by:

$$\sum_{i=0}^{t} a_t^i \alpha_t^i = \sum_{i=0}^{t-1} a_{t-1}^i [(i+1)\alpha_t^i + (t-i)\alpha_t^{i+1}] \quad (59)$$

Formula shall be demonstrated by applying the recurrence coefficients a_t^i and re-index of terms. Relationship is independent of the defining coefficients α_t^i, however, require the same type of index after i for them A direct application of this relationship is:

$$\sum_{i=0}^{t} a_t^i s_t(n-i) = \sum_{i=0}^{t-1} a_{t-1}^i [(i+1)s_t(n-i) + (t-i)s_t(n-i-1)] \quad (60)$$

In principle lower the in coefficient system a_t^i can be applied between any two consecutive levels t and t-1 without upper level to the maximum defined t. The application is defined to value t=1 into relation

Property 5: A defined sum of type $\sum_{i=0}^{t} a_t^i \alpha_t^i$ at level t has image at all lower levels in the system of a_t^i coefficients.

This will write:

$$\sum_{i=0}^{t} a_t^i \alpha_t^i = \sum_{i=0}^{f} a_f^i \; {}^{f}\beta_t^i \;\; \forall\; 0 \le f \le t-1 \;\; f \in N \quad (61)$$

Regardless of level f to which descends in a_t^i coefficient system values ${}^{f}\beta_t^i$ are defined at t level, they are based on the values α_t^i defined at the t level.

Property 6: By lowering in the system of a_t^i coefficients transfer defining the type of procedure values to values at lower levels ${}^{f}\beta_t^i$.

Demonstration:

A check between the t and t-1 levels is sufficient, the calculation between the other levels take places similarly. Either:

$${}^{t-1}\beta_t^i = (i+1)\alpha_t^i + (t-i)\alpha_t^{i+1}$$

Then:

$$[(i+2)\alpha_t^{i+1} + (t-i-1)\alpha_t^{i+2}] =$$

$${}^{t-1}\beta_t^i - {}^{t-1}\beta_t^{i+1} = [(i+1)\alpha_t^i + (t-i)\alpha_t^{i+1}] -$$

$$(i+1)(\alpha_t^i - \alpha_t^{i+1}) + (t-i-1)(\alpha_t^{i+1} - \alpha_t^{i+2})$$

$$+\alpha_t^{i+1} - \alpha_t^{i+1} =$$

$$[(i+2)\alpha_t^{i+1} + (t-i-1)\alpha_t^{i+2}] =$$

$$(i+1)\alpha_{t-1}^i + (t-i-1)\alpha_{t-1}^{i+1} = {}^{t-1}\beta_{t-1}^i$$

Is obtained:

$${}^{t-1}\beta_t^i - {}^{t-1}\beta_t^{i+1} = {}^{t-1}\beta_{t-1}^i \quad (62)$$

Exponent left is part of the the name β coefficients will not change with of t variabile. It looks like that the same type of procedure (58) associated with one level will send lower levels .For α_t^i coefficients their recurrence relation to leads to the possibility express them based on values $\alpha_p^0 \;\; 0 \le p \le t$.

So these are define at t level, by (t+1) values indexed by i or (t+1) values indexed after p at i=0. The transition between the two expressions is given by:

$$\alpha_t^i = \sum_{p=0}^{i} (-1)^p C_i^p \alpha_{t-p}^0 \quad (63)$$

$0 \le p \le i \;\; 0 \le i \le t \;\; p,i,t \in N$

It shows that increasing values of i is equivalent to lowering of p at i = 0.

9. STUDY OF RELATION $\sum_{i=0}^{t} a_t^i \alpha_t^i = 0$

The values α_t^i are defined by a calculation process applied to the t level. In this discussion is not considered the method calculation it is important that it exists and the following take place:

1. Recurrence takes place:

$$\alpha_p^i - \alpha_p^{i+1} = \alpha_{p-1}^i \;\; 0 \le p \le t,\; p \in N \quad (64)$$

2. $\alpha_t^i \in Z$ (are integers) (65)

Property 7: The relation $\sum_{i=0}^{t} a_t^i \alpha_t^i = 0$ (66)

occurs if and only if: $\alpha_t^i = A_t^i \;\; 0 \le i \le t \;\; i,t \in N$ (67)

$$A_t^i = \sum_{j=1}^{t} w_t^j R_{i,t}^j \quad (68)$$

Motivation: Determination unique of a_t^i coefficients in the system.of coefficients $R_{i,t}^j$

Values A_t^i are determined by coefficients w_t^j.

It is important to determine whether they are rational or irrational. We define a decreasing procedure similar to the values α_t^i for values A_t^i starting from the t level. This is given by the relations:

$$A_t^i - A_t^{i+1} \stackrel{def}{=} A_{t-1}^i \quad (69)$$

$$A_p^i - A_p^{i+1} \stackrel{def}{=} A_{p-1}^i \; 0 \le p < t,\; p \in N \quad (70)$$

Property 8: Defined values A_p^i have the following calculation relation:

$$A_p^i = \frac{(t+1)!}{(p+1)!} \sum_{j=0}^{p} w_t^{j+t-p} R_{i,p}^j \; 0 \le p < t,\, p \in N \quad (80)$$

We show the calculation for A_{t-1}^i

$$A_{t-1}^i = A_t^i - A_i^{i+1} = \sum_{j=1}^{t} w_t^j (R_{i,t}^j - R_{i+1,t}^j) =$$

$$\sum_{j=1}^{t} (t+1)! w_t^j R_{i,t-1}^{j-1} \underset{j-1 \to j}{=} (t+1) \sum_{j=0}^{t-1} (t+1)! w_t^{j+1} R_{i,t-1}^j \quad (81)$$

The formula obtained for lower levels is not assimilated directly into the t level. For this missing term j = 0 (constant term).

Property 9: We have successive implications:

$$1) \sum_{i=0}^{t} a_t^i \alpha_t^i = 0; \quad (66)$$

$$2)\; \alpha_t^i = A_t^i \; 0 \le i \le t \; i,t \in N \quad (67)$$

$$3)\; \alpha_p^i = A_p^i \; 0 \le i \le p \; 0 \le p \le t,\; p \in N \; i,t \in N \quad (82)$$

4) Coefficients w_t^j are rational and (t +1)! is a common multiple of their denominators. (83)

To statement 3 applies to the decrease in procedure both for α_p^i values and A_p^i values.

To calculate:

$$\sum_{i=0}^{p} a_p^i \alpha_p^i =$$

$$\sum_{i=0}^{p} a_p^i A_p^i =$$

$$\sum_{i=0}^{p} a_p^i \frac{(t+1)!}{(p+1)!} \sum_{j=0}^{p} w_t^{j+t-p} R_{i,p}^j =$$

$$\sum_{j=0}^{p} \frac{(t+1)!}{(p+1)!} w_t^{j+t-p} \sum_{i=0}^{p} a_p^i R_{i,p}^j =$$

$$+ \frac{(t+1)!}{(p+1)!} w_t^{t-p} \sum_{i=0}^{p} a_p^i R_{i,p}^0$$

$$+ \sum_{j=1}^{p} \frac{(t+1)!}{(p+1)!} w_t^{j+t-p} \sum_{i=0}^{p} a_p^i R_{i,p}^j =$$

$$+ \sum_{j=1}^{p} \frac{(t+1)!}{(p+1)!} w_t^{j+t-p}\, 0 =$$

$$+ \frac{(t+1)!}{(p+1)!} w_t^{t-p} (p+1)! = (t+1)!\, w_t^{t-p} \quad (84)$$

Obtain the relation:

$$\sum_{i=0}^{p} a_p^i \alpha_p^i = (t+1)!\, w_t^{t-p} \quad (83)$$

The sum of of the left is an integer from the definition of values α_t^i (supported by a separate calculation process). By applying decrease in procedure will get for all α_p^i integer values. It also requires the right member to be integer. Thus was obtained statement 4).

By lowering the system of a_t^i coefficients of the relation $\sum_{i=0}^{t} a_t^i \alpha_t^i = 0$ are obtained identically zero relations that do not provide any additional information.

$$\sum_{i=0}^{t} a_t^i \alpha_t^i = \sum_{i=0}^{t-1} a_{t-1}^i [(i+1)\alpha_t^i + (t-i)\alpha_t^{i-1}] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^i \sum_{j=1}^{t} w_t^j [(i+1) R_{i,t}^j + (t-i) R_{i+1,t}^j] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^i \sum_{j=1}^{t} w_t^j [(i+1)(R_{i,t}^j - R_{i+1,t}^j) + (t+1) R_{i+1,t}^j] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^i \sum_{j=1}^{t} w_t^j [(i+1)(t+1) R_{i,t-1}^{j-1} + (t+1) R_{i+1,t}^j] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^i \sum_{j=1}^{t} w_t^j (t+1)[-(-i-1) R_{i,t-1}^{j-1} + R_{i+1,t}^j] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^i \sum_{j=1}^{t} w_t^j (t+1)[-R_{i+1,t}^j + R_{i+1,t}^j] =$$

$$\sum_{i=0}^{t-1} a_{t-1}^{i} \cdot 0 = 0 \qquad (85)$$

Property 10: The existence a relation type

$$\sum_{i=0}^{t} a_t^i \alpha_t^i = 0$$

the same type relations will be generates at t level.

By applying the recurrence relation $\alpha_t^i - \alpha_t^{i+1} = \alpha_{t-1}^i$ is obtained:

$$\sum_{i=0}^{t} a_t^i (\alpha_{t-1}^i + \alpha_t^{i+1}) = 0 \qquad (86)$$

This relation is the same type as the original because it respects the the relation:

$$(\alpha_{t-1}^i + \alpha_t^{i+1}) - (\alpha_{t-1}^{i+1} + \alpha_t^{i+2}) = \alpha_{t-2}^i + \alpha_{t-1}^{i+1} \qquad (87)$$

By generalization of method is obtained relations type:

$$\sum_{i=0}^{t} a_t^i \sum_{k=0}^{f} C_f^k \alpha_{t-f+k}^{i+k} = 0 \quad f \in N \qquad (88)$$

f is a free variable thus generate an infinite number of relations same type as the original which will be each of type A_t^i It was shown that regardless of the which relation (depending on f) start at t=0 we obtain the same relation and the for t=1, we obtain a relation independent of f. Relations given by the formula (88) combines levels of α_t^i procedure

CONCLUSIONS

Developments indicate a particular algebraic field. Therefore concepts were introduced (classification of procedures after t):

- direct defined procedure (E.g. natural procedure);
- increasing recurrence defined procedure E.g.: a_t^i;
- decreasing defined procedure Eg, ${}^f\beta_t^i$, A_t^i, the levels t=0, and t=-1 of the natural procedure;
- recursively defined procedure and inputs at one level (given by the variable n or t) E.g:

Upper natural procedure and Right natural procedure, procedure type α_t^i;

- procedures defined as an input at each level. E.g.: α_p^i for i=0.

In the paper were present relations that show functionality of algebraic system.

The main result obtained in this paper is given by property 9. Particular characterizations of the relations of type (66) are given by relations (85) and (88).

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