EXPONENTIATED POWER QUASI LINDLEY DISTRIBUTION. SUBMODELS AND SOME PROPERTIES

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Abstract: *In this paper we introduce a new generalization of the Lindley distribution which generalizes the power Lindley distribution, proposed by Ghitany, and another form of generalized Lindley proposed by Nadarajah. This new kind of generalized Lindley distribution has four parameters and it allows more adaptability to analyze real lifetime data.*

Keywords: *Lindley distribution, order statistics*

1. INTRODUCTION

In this paper, we deal with a generalization of Lindley distribution because it forms a flexible family of distributions with an important selection of shape and hazard functions. The Lindley distribution was firstly proposed by Lindley (1958) in the context of Bayesian statistics, based on Bayes theorem [1], [2] as a counterexample of fiducial statistics. Mixing various distributions lead to the expansion of known families of distributions. In literature, there were introduced and studied some mixed data modeling distributions of life as Weibull Poisson, Weibull geometric, Exponential geometric.

Lindley distribution is a one-parameter distribution, given by its probability density function

$$f(x)=\frac{\theta^2}{\theta+1}(x+1)e^{-\theta x} \tag{1}$$

The cumulative distribution function corresponding to (1) is

$$F(x)=1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}, \quad x>0, \theta>0. \tag{2}$$

The properties of the Lindley distribution were studied by M.E. Ghitany, B. Atieh, S. Nadarajah [4, 5, 6]. They discussed its applications to survival data and, also, showed in a numerical example that the Lindley distribution gives better modeling for waiting times and survival time data than the exponential distribution. Different forms of generalized Lindley distributions were been widely applied for reliability modeling and life testing data [10, 11]. There is a great development of another various quantitative techniques for solving optimization problems for biological and economical domains [8, 9].

Definition 1.1 Let X be a random variable and the parameters $\alpha, \theta > 0$. We say that X has a quasi Lindley distribution $X \sim QL(\alpha, \theta)$ if it has the probability density function

$$f(x) = \frac{\theta^2}{\alpha\theta + 1} e^{-\theta x} (\alpha + x)$$

and the cumulative distribution function

$$F(x) = 1 - \frac{\alpha\theta + 1 + \theta x}{\alpha\theta + 1} e^{-\theta x}, \quad x > 0, \alpha, \theta > 0$$

Because the Lindley distribution (having only one parameter) does not provide enough flexibility for analyzing different types of lifetime data, in statistic literature it were introduced some new compounding the Lindley distribution with Negative Binomial distribution [3], with Poisson distribution [4] or Exponential Poisson [7] offering some new distributions of lifetime case obtaining from Generalized Lindley distribution compounding with exponential and gamma distributions. The quasi Lindley distribution reduces of the one following known distribution:

1. For $\alpha = 1$, it becomes $Lindley(\theta)$
2. For $\alpha = 0$, it becomes $Gamma(2, \theta)$
3. For $\alpha \to \infty$, it becomes $Exp(\theta)$

The quasi Lindley distribution maybe can write as a two-component mixture of $Exp(\theta)$ and $Gamma(2, \theta)$:

$$f(x) = pf_1(x) + (1-p) f_2(x), \quad x > 0, \alpha, \theta > 0$$

where $p = \frac{\alpha\theta}{\alpha\theta + 1}$, $f_1(x) = \theta e^{-\theta x}$ and $f_2(x) = \theta^2 x e^{-\theta x}$.

Ghitany et al. proposed the power transformation, $Y = X^{1/\beta}, \beta > 0$, for Lindley distribution for generating a flexible family of probability distributions. The new parameter would offer more distributional flexibility with a form of the hazard rate what can be decreasing, unimodal and decreasing-increasing-decreasing for some particular cases of the parameters.

Definition 1.2 Let X be a random variable and the parameters $\alpha, \theta, \beta > 0$. We say that X has a quasi-power Lindley distribution $X \sim QPL(\alpha, \theta, \beta, b)$ if it has the probability density function

$$f(y) = \frac{\alpha\theta^2\beta}{\alpha\theta + 1} y^{\beta-1} e^{-\theta y^\beta} + \frac{\beta\theta^2}{\alpha\theta + 1} y^{2\beta-1} e^{-\theta y^\beta}$$

and the cumulative distribution function

$$F(y) = 1 - \left(1 + \frac{\theta y^\beta}{\alpha\theta + 1}\right) e^{-\theta y^\beta}, \quad y > 0, \alpha, \theta, \beta > 0$$

The quasi power Lindley distribution may be can write as a two-component mixture of $Weibull(\beta,\theta)$ and $Gamma(2,\beta,\theta)$:

$$f(y)=pf_1(y)+(1-p)f_2(y), \quad y>0, \alpha, \theta, \beta>0$$

where $p=\dfrac{\alpha\theta}{\alpha\theta+1}$, $f_1(y)=\theta\beta y^{\beta-1}e^{-\theta y^{\beta}}$ and $f_2(y)=\theta^2\beta y^{2\beta-1}e^{-\theta y^{\beta}}$.

We introduce a new four parameter distribution, denoted

$X \sim EQPL(\alpha,\theta,\beta,b), \alpha,\theta,\beta,b>0$

referred to as the exponentiated quasi power Lindley. This new distribution reduces to the quasi Lindley distribution, the exponential distribution and gamma distribution. On terms of reliability, the various shapes of the EQPL distribution give it an advantage, being more suitable to model many real systems which generally exhibit bath-tub shaped failure rate.

Definition 1.3 Let $X \sim EQPL(\alpha,\theta,\beta,b)$. The cumulative function of the $EQPL(\alpha,\theta,\beta,b)$, $\alpha,\theta,\beta,b>0$ is

$$F(x)=\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b} \tag{3}$$

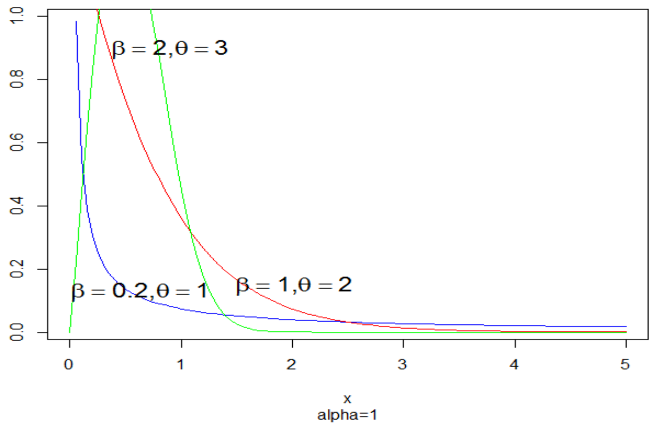
and the corresponding probability density is given by

$$f(x)=\frac{\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}, \quad x>0 \tag{4}$$

The EQPL distribution reduces of the one following known distribution:

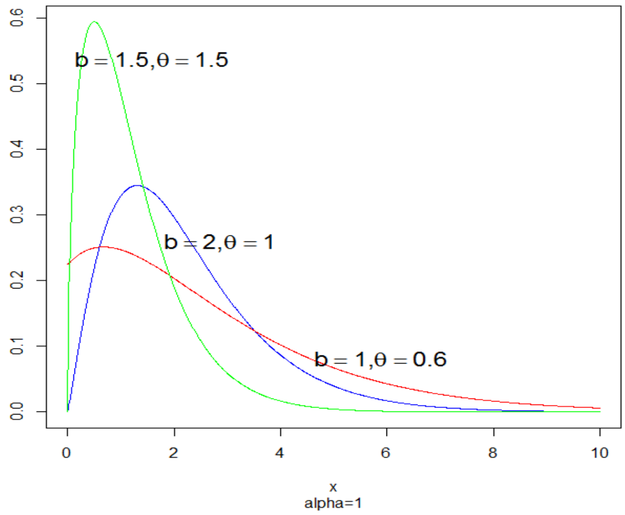
1. For b=1, it becomes $QPL(\alpha,\theta,\beta)$

$$f(x,\alpha,\theta,\beta)=\frac{\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}$$



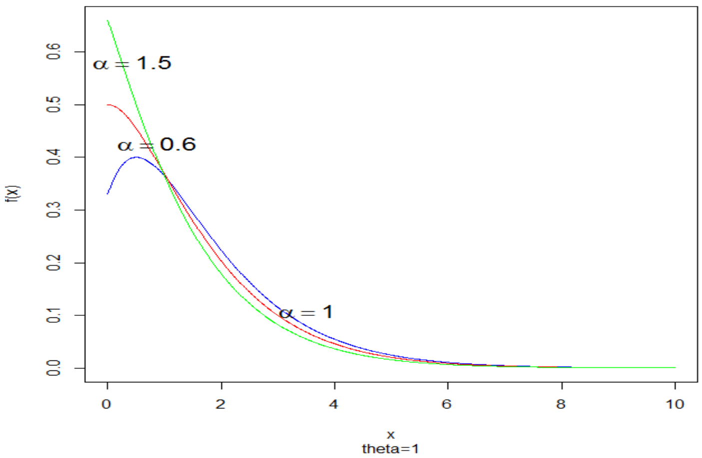
2. For $\beta = 1$, it becomes PowerLindley (α, θ, b)

$$f(x,\alpha,\theta,b) = b\theta^2 e^{-\theta x} \frac{\alpha + x}{\theta\alpha + 1}\left(1 - e^{-\theta x}\left(\frac{\theta x}{\theta\alpha + 1} + 1\right)\right)^{b-1}$$

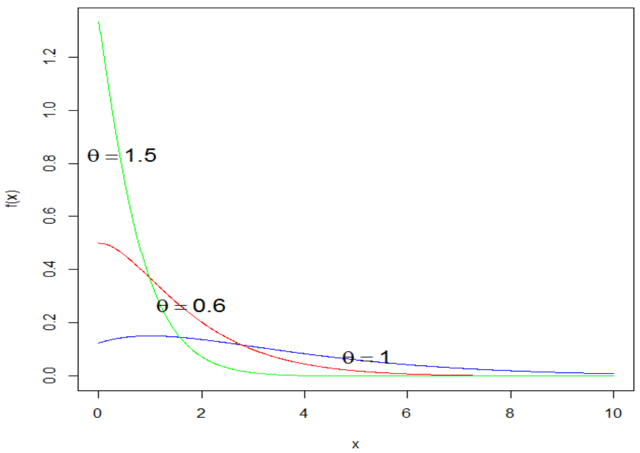


3. For $b = \beta = 1$, it becomes $QL(\alpha, \theta)$

$$f(x,\alpha,\theta) = \theta^2 e^{-\theta x} \frac{\alpha + x}{\theta\alpha + 1}$$



4. For $b = \beta = \alpha = 1$, it becomes $L(\theta)$



The cdf of X can also be represented in an extended form

$$F(x)=\sum_{i=1}^{\infty}\binom{b}{i}p^{b-i}(1-p)^{i}F_{W(\beta,\theta)}^{b-i}(x)F_{GGamma(2,\beta,\theta)}^{i}(x)$$

Definition 1.4 The corresponding hazard rate function is

$$h(x)=\frac{\beta\theta^{2}b}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\left\{S(x)\right\}^{-1}$$

where

$$S(x)=S(x,\alpha,\theta,\beta,b)=1-\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}$$

2. STOCHASTIC ORDER

Let $X_1 \sim EQPL(\alpha_1,\theta_1,\beta_1,b_1)$ and $X_2 \sim EQPL(\alpha_2,\theta_2,\beta_2,b_2)$ be two exponentiated quasi power Lindley random variables with common shape β. Let F1 denote the cumulative distribution function of X_1 and F_2 the cumulative distribution function for X_2.

Defnition 2.1 We say that X_1 is stochastically greater or equal than $X_2\left(X_1 \geq_{st} X_2\right)$ if $F_{X_1}(x)\leq F_{X_2}(x)$, for all x where $F_{X_1}(x)$ and $F_{X_2}(x)$ are the cdfs of X_1 and X_2, respectively.

Definition 2.2 We say that X_1 is stochastically greater than X_2 with respect to likelihood ratio $\left(X_1 \geq_{lr} X_2\right)$ if $\frac{f_{X_2}(x)}{f_{X_1}(x)}$ is an increasing function of x, where $F_{X_1}(x)$ and $F_{X_2}(x)$ are the cdfs of X_1 and X_2, respectively.

Definition 2.3 We say X_1 is stochastically greater than X_2 with respect to reverse hazard rate $\left(X_1 \geq_{hr} X_2\right)$ if $h_{X_1}(x)\leq h_{X_2}(x)$ and for all x.

For establishing stochastic order we have the following important results due to Shaked and Shantikumar

$$\left(X_1 \geq_{lr} X_2\right)\Rightarrow\left(X_1 \geq_{hr} X_2\right)\Rightarrow\left(X_1 \geq_{st} X_2\right)$$

The EQPL distribution is ordered with respect to the strongest one as shown in the following theorem.

Also, we have

$$\frac{f_{X_2}(x)}{f_{X_1}(x)}=\frac{\beta_2\theta_2^2 b_2 x^{\beta_2-1}\left(\alpha_2+x^{\beta_2}\right)}{\beta_1\theta_1^2 b_1 x^{\beta_1-1}\left(\alpha_1+x^{\beta_1}\right)}e^{-\theta_2 x^{\beta_2}+\theta_1 x^{\beta_1}}\times\left[1-\left(1+\frac{\theta_2 x^{\beta_2}}{\alpha_2\theta_2+1}\right)e^{-\theta_2 x^{\beta_2}}\right]^{b_2-1}\times$$

$$\times\left[1-\left(1+\frac{\theta_1 x^{\beta_1}}{\alpha_1\theta_1+1}\right)e^{-\theta_1 x^{\beta_1}}\right]^{1-b_1} \tag{5}$$

Theorem 2.4 If we have $\alpha_1=\alpha_2, \theta_1=\theta_2$ and $\beta_1=\beta_2$ then X_2 is stochastically greater with respect to likelihood ratio than X_1 if and only if $b_1>b_2$.

Theorem 2.5 If we have $\alpha_1=\alpha_2, b_1=b_2=b\geq 1$ and $\beta_1=\beta_2$ then X_2 is stochastically greater with respect to likelihood ratio than X_1 if and only if $\theta_2\geq\theta_1$.

Theorem 2.6 If we have $\alpha_1=\alpha_2, b_1=b_2=b<1$ and $\beta_1=\beta_2$ then X_2 is stochastically greater with respect to likelihood ratio than X_1 if and only if $\theta_2\leq\theta_1$.

3. MOMENTS

We will obtain the moments of the EQPL distribution using the binomial series expansion.

Theorem 3.1 The rth moment of the exponentiated quasi power Lindley $E\left(X^r\right)$ is given by

$$E\left(X^r\right)=C_{i,k}\frac{\beta\theta^2}{\alpha\theta+1}\left[\alpha\frac{\Gamma\left(\frac{r+bk}{\beta}+1\right)}{\left[\theta(1+i)\right]^{\frac{r+bk}{\beta}+1}}+\alpha\frac{\Gamma\left(\frac{r+bk}{\beta}+2\right)}{\left[\theta(1+i)\right]^{\frac{r+bk}{\beta}+2}}\right]$$

Proof.

$$E\left(X^r\right)=\int_0^\infty x^r f(x)dx,$$

$$E\left(X^r\right)=\int_0^\infty\frac{\beta\theta^2}{\alpha\theta+1}x^{r+\beta-1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}dx=$$

$$=\int_0^\infty\frac{\alpha\beta\theta^2 b}{\alpha\theta+1}x^{r+\beta-1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}+$$

$$+\int_0^\infty\frac{\beta\theta^2 b}{\alpha\theta+1}x^{r+2\beta-1}e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}dx.$$

Using the binomial series expansion of $\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}$ given by

$$\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}=\sum_{i=0}^{\infty}C_{b-1}^{i}(-1)^{i}\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)^{i}e^{-i\theta x^{\beta}}$$

and using the following binomial series expansion of $\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)^{i}$ given by

$$\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)^{i}=\sum_{k=0}^{\infty}C_{i}^{k}\left(\frac{\theta}{\alpha\theta+1}\right)^{k}x^{k\beta}$$

We obtain the rth moment of X and we find

$$E\left(X^{r}\right)=C_{i,k}\int_{0}^{\infty}\frac{\alpha\beta\theta^{2}b}{\alpha\theta+1}x^{r+\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}dx+C_{i,k}\int_{0}^{\infty}\frac{\beta\theta^{2}b}{\alpha\theta+1}x^{r+2\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}dx.$$

Let $y=x^{\beta}$ and

$$C_{i,k}=\sum_{i=0}^{\infty}C_{b-1}^{i}(-1)^{i}\sum_{i=0}^{\infty}C_{i}^{k}\left(\frac{\theta}{\alpha\theta+1}\right)^{k}.$$

So, the rth moment can be rewritten

$$E\left(X^{r}\right)=C_{i,k}\frac{\beta\theta^{2}}{\alpha\theta+1}\left[\alpha\frac{\Gamma\left(\frac{r+bk}{\beta}+1\right)}{\left[\theta(1+i)\right]^{\frac{r+bk}{\beta}+1}}+\alpha\frac{\Gamma\left(\frac{r+bk}{\beta}+2\right)}{\left[\theta(1+i)\right]^{\frac{r+bk}{\beta}+2}}\right]$$

Theorem 3.2 The moment generating function of the exponentiated quasi power Lindley $M_X(t)$ is given by

$$M_X(t)=C_{i,k,j}\frac{\beta\theta^{2}}{\alpha\theta+1}\left[\alpha\frac{\Gamma\left(\frac{k\beta+j}{\beta}+1\right)}{\left[\theta(1+i)\right]^{\frac{k\beta+j}{\beta}+1}}+\alpha\frac{\Gamma\left(\frac{k\beta+j}{\beta}+2\right)}{\left[\theta(1+i)\right]^{\frac{k\beta+j}{\beta}+2}}\right]$$

Proof.

$$M_X(t)=\int_0^{\infty} e^{tx} f(x)\,dx$$

$$M_X(t)=\int_0^{\infty} e^{tx}\frac{\beta\theta^2}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}dx=$$

$$=\int_0^{\infty}\frac{\alpha\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1}e^{-\theta x^{\beta}}e^{tx}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}+$$

$$+\int_0^{\infty}\frac{\beta\theta^2 b}{\alpha\theta+1}x^{2\beta-1}e^{-\theta x^{\beta}}e^{tx}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}dx.$$

Using the binomial series expansion like in the last theorem, we introduce the last two expansions in the moment generating function of X and we find

$$M_X(t)=C_{i,k}\int_0^{\infty}\frac{\alpha\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}e^{tx}dx+C_{i,k}\int_0^{\infty}\frac{\beta\theta^2 b}{\alpha\theta+1}x^{2\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}e^{tx}dx.$$

Let consider

$$e^{tx}=\sum_{=0}^{\infty}\frac{t^j x^j}{j!}.$$

So, the moment generating function can be rewritten

$$M_X(t)=C_{i,k}\sum_{=0}^{\infty}\frac{t^j}{j!}(\int_0^{\infty}\frac{\alpha\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}e^{tx}dx+C_{i,k}\int_0^{\infty}\frac{\beta\theta^2 b}{\alpha\theta+1}x^{2\beta-1+k\beta}e^{-\theta(1+i)x^{\beta}}e^{tx})dx.$$

Let $y=x^{\beta}$

$$C_{i,k,j}=C_{i,k}\sum_{=0}^{\infty}\frac{t^j}{j!}.$$

The moment generating function has the last form

$$M_X(t)=C_{i,k,j}\frac{\beta\theta^2}{\alpha\theta+1}\left[\alpha\frac{\Gamma\left(\frac{k\beta+j}{\beta}+1\right)}{\left[\theta(1+i)\right]^{\frac{k\beta+j}{\beta}+1}}+\alpha\frac{\Gamma\left(\frac{k\beta+j}{\beta}+2\right)}{\left[\theta(1+i)\right]^{\frac{k\beta+j}{\beta}+2}}\right].$$

4. GENERATION ALGORITHMS

We consider simulating values of a random variable $X \sim EQPL(\alpha, \lambda, \beta, b)$

Algorithm 1

1. Generate $U_i \sim U(0,1), i = \overline{1,n}$
2. Set $X_i = \left\{ -\alpha - \frac{1}{\theta} - \frac{1}{\theta} W\left[-(\alpha\theta + 1)\left(1 - U_i^{1/b}\right) \exp(-\alpha\theta - 1) \right] \right\}^{1/\beta}, \ i = \overline{1,n}.$

Algorithm 2

1. Generate $U_i \sim U(0,1), i = \overline{1,n}$
2. Generate $V_i \sim Exponential(0), i = \overline{1,n}$
3. Generate $G_i \sim \Gamma(2,0), i = \overline{1,n}$
4. If $U_i^{1/\beta} < p$, $p = \frac{\alpha\theta}{\alpha\theta + 1}$, then set $X_i = V_i^{1/\beta}$ otherwise $X_i = G_i^{1/\beta}, i = \overline{1,n}$.

Simulation study

```
n=10
theta=seq(0,4,length=10)
beta=seq(0,4,length=10)
alpha=seq(0,4,length=10)
u=runif(n)
v=rexp(theta)
g=rgamma(2*beta,theta)
p=(alpha*theta)/(alpha*theta+1)
if (u^(1/b)<p)
{x=v^(1/b)}
else {x=g^(1/b)
x
[1] 0.86096886 0.02832007 0.89052860 0.16637737 0.27854706 0.78316390
[7] 0.30347644 0.42973307 0.21931648 0.76245404
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