CUMULATIVE ENTROPIES: A SURVEY

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Abstract: *We present a review of cumulative entropies from reliability theory.*
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1. INTRODUCTION

Consider X a non-negative absolutely continuous random variable with the probability density function (PDF) $f(x)$, the cumulative distribution function (CDF) $F(x) = P(X \le x)$, and the reliability function (RF) $\overline{F}(x) = P(X > x)$.

Let $t \in [0,\infty)$ and w a real nonnegative measurable function defined on $[0,\infty)$.

Let $D = \{(t_1,t_2) \mid 0 \le t_1 < t_2 \le \infty, F(t_1) < F(t_2)\}$.

For any given pair $(t_1,t_2) \in D$, the conditional random variable $(X \quad t_1 \le X \le t_2)$ has the PDF $f(x;t_1,t_2)$, the CDF $F(x;t_1,t_2)$ and the RF $\overline{F}(x;t_1,t_2)$ given by

$$
\begin{aligned}
&f(x;t_1,t_2) = \frac{f(x)}{F(t_2)-F(t_1)}, \; F(x;t_1,t_2) = \frac{F(x)}{F(t_2)-F(t_1)}, \\
&\overline{F}(x;t_1,t_2) = \frac{\overline{F}(x)}{F(t_2)-F(t_1)} = \frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)}, \text{ for all } t_1 \le x \le t_2.
\end{aligned}
\tag{1}
$$

The **Shannon entropy** proposed by Shannon [45] (1948) is defined as

$$
H(X) = -\int_0^\infty f(x)\ln f(x)dx. \tag{2}
$$

Remark 1.1 *In all definition from this paper we assume that the integrals exist, and we use the notational convention* $0\ln 0 = 0$ *and* $0/0 = 0$.

The concept of entropy was proposed as a measure of the amount of information supplied by a random variable X or a probabilistic experiment. It has numerous extensions to other entropy-type measures, some of these will be presented below.

In the next sections we present the residual and past entropy-types measures based on CDF and RF, the cumulative entropies, the cumulative relative entropies and inaccuracy measures.

2. WEIGHTED, RESIDUAL AND PAST ENTROPIES

The **weighted entropy**, referred by Di Crescenzo and Longobardi [9] (2006) in agreement with Beli s and Guia s u [4] (1968), is defined as

$$H_w(X) = -\int_0^\infty w(x) f(x) \ln f(x) dx. \tag{3}$$

The **residual entropy (RE)** proposed by Ebrahimi and Pellerey [13] (1996) is defined as

$$RE(X;t) = -\int_t^\infty f(x;t,\infty) \ln f(x;t,\infty) dx = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \ln \frac{f(x)}{\overline{F}(t)} dx. \tag{4}$$

The **past entropy (PE)** proposed by Di Crescenzo and Longobardi [7] (2002) is defined as

$$PE(X;t) = -\int_0^t f(x;0,t) \ln f(x;0,t) dx = -\int_0^t \frac{f(x)}{F(t)} \ln \frac{f(x)}{F(t)} dx. \tag{5}$$

The **weighted residual entropy (WRE)** proposed by Di Crescenzo and Longobardi [9] (2006) is defined as

$$WRE(X;t) = -\int_t^\infty x\, f(x;t,\infty) \ln f(x;t,\infty) dx = -\int_t^\infty x \frac{f(x)}{\overline{F}(t)} \ln \frac{f(x)}{\overline{F}(t)} dx. \tag{6}$$

The **weighted past entropy (WPE)** proposed by Di Crescenzo and Longobardi [9] (2006) is defined as

$$WPE(X;t) = -\int_0^t x\, f(x;0,t) \ln f(x;0,t) dx = -\int_0^t x \frac{f(x)}{F(t)} \ln \frac{f(x)}{F(t)} dx. \tag{7}$$

These weighted entropies are suitable to describe dynamic information of random lifetimes, in analogy with the entropies of residual and past lifetimes introduced in [13] and [7], respectively.

3. CUMULATIVE ENTROPIES

The **cumulative residual entropy (CRE)** proposed by Rao, Chen, Vemuri, Wang [41] (2004) is defined as

$$CRE(X) = -\int_0^\infty \overline{F}(x) \ln \overline{F}(x) dx. \tag{8}$$

The CRE is an alternative measure of uncertainty in the random variable X that enjoys many of the properties of Shannon entropy and has some advantages such us is always non-negative, can be easily computed from sample data and these computations asymptotically converge to the true values.

Like as the Shannon entropy the CRE can be used to construct probability distributions by applying the Maximum Entropy Principle introduced in 1957 by Jaynes [18, 19]. For example, Rao [42] (2005) obtains a general result for characterization of MAX-CRE distributions and applies this result to construct the uniform distribution and the Weibull distribution.

Drissi, Chonavel and Boucher [12] (2008) generalize the definition of CRE to the case of random variables with supports that are not restricted to positive values.

The **dynamic cumulative residual entropy (DCRE)** proposed by Asadi and Zohrevand [2] (2007) is defined as

$$DCRE(X;t) = -\int_t^{\infty} \overline{F}(x;t,\infty)\ln \overline{F}(x;t,\infty)dx = -\int_t^{\infty} \frac{\overline{F}(x)}{\overline{F}(t)}\ln\frac{\overline{F}(x)}{\overline{F}(t)}dx. \tag{9}$$

The DCRE is a measure of the information in the residual life distribution. The authors show that the CRE and the DCRE is connected with some well-known reliability measures such as the mean residual lifetime and the hazard rate. Also, they prove that if the $DCRE(X;t)$ is an non-decreasing function on t then it characterizes the underlying distribution function uniquely.

The **cumulative past entropy (CPE)** proposed by Di Crescenzo and Longobardi [10] (2009) is defined as

$$CPE(X) = -\int_0^{\infty} F(x)\ln F(x)dx. \tag{10}$$

The CPE is also non-negative and it is useful to measure information on the inactivity time of a system, being appropriate for the systems whose uncertainty is related to the past.

The **dynamic cumulative past entropy (DCPE)** proposed by Di Crescenzo and Longobardi [10] (2009) and by Navarro, Del Aguila and Asadi [37] (2010) is defined as

$$DCPE(X;t) = -\int_0^{t} F(x;0,t)\ln F(x;0,t)dx = -\int_0^{t} \frac{F(x)}{F(t)}\ln\frac{F(x)}{F(t)}dx. \tag{11}$$

The **interval entropy (IH)** proposed by Sunoj, Sankaran and Maya [47] (2009) is defined as

$$IH(X;t_1,t_2) = -\int_{t_1}^{t_2} f(x;t_1,t_2)\ln f(x;t_1,t_2)dx = -\int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)}\ln\frac{f(x)}{F(t_2)-F(t_1)}dx. \tag{12}$$

The **weighted cumulative residual entropies (WCRE)** proposed by Misagh, Panahi, Yari, Shahi [33] (2011) is defined as

$$WCRE(X) = -\int_0^{\infty} x\,\overline{F}(x)\ln \overline{F}(x)dx. \tag{13}$$

The **weighted cumulative past entropies (WCPE)** proposed by Misagh, Panahi, Yari, Shahi [33] (2011) is defined as

$$WCPE(X) = -\int_0^{\infty} x\,F(x)\ln F(x)dx. \tag{14}$$

The authors present various properties of this measure, including its connection with weighted residual and past entropies and obtain some upper and lower bounds.

The **interval cumulative residual entropies (ICRE)** proposed by Khorashadizadeh, Rezaei Roknabadi and Mohtashami Borzadaran [25] (2013) is defined as

$$ICRE(X;t_1,t_2) = -\int_{t_1}^{t_2} \overline{F}(x;t_1,t_2) \ln \overline{F}(x;t_1,t_2) dx = -\int_{t_1}^{t_2} \frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)} \ln \frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)} dx. \quad (15)$$

The **interval cumulative past entropies (ICPE)** proposed by Khorashadizadeh, Rezaei Roknabadi and Mohtashami Borzadaran [25] (2013) is defined as

$$ICPE(X;t_1,t_2) = -\int_{t_1}^{t_2} F(x;t_1,t_2) \ln F(x;t_1,t_2) dx = -\int_{t_1}^{t_2} \frac{F(x)}{F(t_2)-F(t_1)} \ln \frac{F(x)}{F(t_2)-F(t_1)} dx. \quad (16)$$

The authors present some properties and characterization of this measures, including its connections with doubly truncated Shannon entropy and mean residual life.

The **weighted cumulative residual entropies (WCRE)** proposed by Suhov and Yasaei Sekeh [46] (2015) is defined as

$$WCRE_w(X) = -\int_0^{\infty} w(x)\, \overline{F}(x) \ln \overline{F}(x) dx. \quad (17)$$

The **weighted cumulative past entropies (WCPE)** proposed by Suhov and Yasaei Sekeh [46] (2015) is defined as

$$WCPE_w(X) = -\int_0^{\infty} w(x)\, F(x) \ln F(x) dx. \quad (18)$$

The **interval weighted cumulative residual entropy (IWCRE)** of the random variable X at interval $[t_1,t_2]$ with the weight function w is defined by Sekeh, Borzadran and Roknabadi ([51]) (2015)

$$\begin{aligned} IWCRE_w(X;t_1,t_2) &= -\int_{t_1}^{t_2} w(x)\overline{F}(x;t_1,t_2) \ln \overline{F}(x;t_1,t_2) dx \\ &= -\int_{t_1}^{t_2} w(x) \frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)} \ln \frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)} dx. \end{aligned} \quad (19)$$

The **interval weighted cumulative (past) entropy (IWCE)** of the random variable X at interval $[t_1,t_2]$ with the weight function w is defined by Sekeh, Borzadran and Roknabadi ([51]) (2015)

$$\begin{aligned} IWCE_w(X;t_1,t_2) &= -\int_{t_1}^{t_2} w(x)F(x;t_1,t_2) \ln F(x;t_1,t_2) dx \\ &= -\int_{t_1}^{t_2} w(x) \frac{F(x)}{F(t_2)-F(t_1)} \ln \frac{F(x)}{F(t_2)-F(t_1)} dx. \end{aligned} \quad (20)$$

4. CUMULATIVE RELATIVE ENTROPIES AND INACCURACY MEASURES

Consider X and Y two non-negative absolutely continuous random variables with the probability density functions (PDFs) $f(x)$ and $g(y)$, the cumulative distribution functions (CDFs) $F(x) = P(X \le x)$ and $G(y) = P(Y \le y)$, and the reliability functions (RFs) $\overline{F}(x) = P(X > x)$ and $\overline{G}(y) = P(Y > y)$ respectively. Let $t \in [0,\infty)$ and $D = \{(t_1,t_2) \mid 0 \le t_1 < t_2 \le \infty, F(t_1) < F(t_2), G(t_1) < G(t_2)\}$.

For any given pair $(t_1,t_2) \in D$, consider the conditional random variable $(X \; t_1 \le X \le t_2)$ with the PDF $f(x;t_1,t_2)$, the CDF $F(x;t_1,t_2)$ and the RF $\overline{F}(x;t_1,t_2)$ defined in the first section and the conditional random variable $(Y \; t_1 \le Y \le t_2)$ has the PDF $g(y;t_1,t_2)$, the CDF $G(y;t_1,t_2)$ and the RF $\overline{G}(y;t_1,t_2)$ given by

$$\begin{gathered} g(y;t_1,t_2) = \frac{g(y)}{G(t_2) - G(t_1)}, G(y;t_1,t_2) = \frac{G(y)}{G(t_2) - G(t_1)}, \\ \overline{G}(y;t_1,t_2) = \frac{\overline{G}(y)}{G(t_2) - G(t_1)} = \frac{\overline{G}(y)}{\overline{G}(t_1) - \overline{G}(t_2)}, \text{ for all } t_1 \le y \le t_2. \end{gathered} \tag{21}$$

The **relative entropy, Kullback-Leibler divergence, Kullback-Leibler discrimination information** proposed by Kullback and Leibler [27] (1951) is defined as

$$D(X,Y) = \int_0^\infty f(x) \ln \frac{f(x)}{g(x)} dx. \tag{22}$$

Developing the Shannon entropy, the authors have the idea to compare the entropy inside a family of probability measures, instead of considering the entropy corresponding to only one probability measure.

The **Kerridge measure of inaccuracy** proposed by Kerridge [24] (1961) is defined as

$$H(X;Y) = -\int_0^\infty f(x) \ln g(x) dx = D(X,Y) + H(X). \tag{23}$$

The **weighted inaccuracy measure** proposed by Taneja and Tuteja [48] (1986) is defined as

$$WH(X;Y) = -\int_0^\infty x f(x) \ln g(x) dx. \tag{24}$$

The **residual relative entropy** proposed by Ebrahimi and Kirmani, [15] (1996) is defined as

$$RD(X,Y;t) = \int_t^\infty f(x;t,\infty) \ln \frac{f(x;t,\infty)}{g(x;t,\infty)} dx = \int_t^\infty \frac{f(x)}{\overline{F}(t)} \ln \frac{f(x)/\overline{F}(t)}{g(x)/\overline{G}(t)} dx. \tag{25}$$

The **past relative entropy** proposed by Crescenzo and Longobardi, [8] (2004) is defined as

$$PD(X,Y;t) = \int_0^t f(x;0,t) \ln \frac{f(x;0,t)}{g(x;0,t)} dx = \int_0^t \frac{f(x)}{F(t)} \ln \frac{f(x)/F(t)}{g(x)/G(t)} dx. \tag{26}$$

The **dynamic measure of inaccuracy** proposed by Taneja, Kumar and Srivastava [49] (2009) is defined as

$$RI(X,Y;t) = -\int_t^\infty f(x;t,\infty)\ln g(x;t,\infty)dx = -\int_t^\infty \frac{f(x)}{\overline{F}(t)}\ln\frac{g(x)}{\overline{G}(t)}dx \tag{27}$$
$$= RD(X,Y;t) + RE(X;t).$$

The **weighted residual inaccuracy measure** proposed by Kumar and Taneja [48] (2012) is defined as

$$WRI(X,Y;t) = -\int_t^\infty xf(x;t,\infty)\ln g(x;t,\infty)dx = -\int_t^\infty x\frac{f(x)}{\overline{F}(t)}\ln\frac{g(x)}{\overline{G}(t)}dx \tag{28}$$
$$= WRD(X,Y;t) + WRE(X;t),$$

where

$$WRD(X,Y;t) = \int_t^\infty xf(x;t,\infty)\ln\frac{f(x;t,\infty)}{g(x;t,\infty)}dx = \int_t^\infty x\frac{f(x)}{\overline{F}(t)}\ln\frac{f(x)/\overline{F}(t)}{g(x)/\overline{G}(t)}dx. \tag{29}$$

represents the **weighted residual relative entropy**.

The **past inaccuracy measure (PI)** proposed by Kumar and Taneja [48] (2012) is defined as

$$PI(X;t) = -\int_0^t f(x;0,t)\ln g(x;0,t)dx = -\int_0^t \frac{f(x)}{F(t)}\ln\frac{g(x)}{G(t)}dx = PD(X,Y;t) + PE(X;t). \tag{30}$$

The **weighted past inaccuracy measure (WPI)** proposed by Kumar and Taneja [48] (2012) is defined as

$$WPI(X;t) = -\int_0^t xf(x;0,t)\ln g(x;0,t)dx = -\int_0^t x\frac{f(x)}{F(t)}\ln\frac{g(x)}{G(t)}dx \tag{31}$$
$$= WPD(X,Y;t) + WPE(X;t),$$

where

$$WPD(X,Y;t) = \int_0^t xf(x;0,t)\ln\frac{f(x;0,t)}{g(x;0,t)}dx = \int_0^t x\frac{f(x)}{F(t)}\ln\frac{f(x)/F(t)}{g(x)/G(t)}dx. \tag{32}$$

represents the **weighted past relative entropy**.

The **cumulative residual inaccuracy (CRI)** proposed by Taneja and Kumar [50] (2012) is defined as

$$CRI(X,Y) = -\int_0^\infty \overline{F}(x)\ln\overline{G}(x)dx = CRD(X,Y) + CRE(X), \tag{33}$$

where

$$CRD(X,Y) = \int_0^\infty \overline{F}(x)\ln\frac{\overline{F}(x)}{\overline{G}(x)}dx \tag{34}$$

represents the **cumulative residual relative entropy**.

The **dynamic cumulative residual inaccuracy (DCRI)** proposed by Taneja and Kumar [50] (2012) (a version was also introduced by Chamany and Baratpour (2014)) is defined as

$$DCRI(X,Y;t) = -\int_t^\infty \overline{F}(x;t,\infty)\ln \overline{G}(x;t,\infty)dx = -\int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}\ln\frac{\overline{G}(x)}{\overline{G}(t)}dx \tag{35}$$

$$= DCRD(X,Y;t) + DCRE(X;t),$$

where

$$DCRD(X,Y;t) = \int_t^\infty \overline{F}(x;t,\infty)\ln\frac{\overline{F}(x;t,\infty)}{\overline{G}(x;t,\infty)}dx = \int_t^\infty \frac{\overline{F}(x)}{\overline{F}(t)}\ln\frac{\overline{F}(x)/\overline{F}(t)}{\overline{G}(x)/\overline{G}(t)}dx \tag{36}$$

represents the **dynamic cumulative residual relative entropy**.

The **interval relative entropy** proposed by Misagh and Yari, [35] (2012) is defined as

$$ID(X,Y;t_1,t_2) = \int_{t_1}^{t_2} f(x;t_1,t_2)\ln\frac{f(x;t_1,t_2)}{g(x;t_1,t_2)}dx$$

$$= \int_{t_1}^{t_2} \frac{f(x)}{F(t_2)-F(t_1)}\ln\frac{f(x)/[F(t_2)-F(t_1)]}{g(x)/[G(t_2)-G(t_1)]}dx. \tag{37}$$

Here was proposed a measure of discrepancy between two lifetime distributions at the interval of time in base of Kullback-Leibler discrimination information. They studied various properties of this measure, including its connection with residual and past measures of discrepancy and interval entropy, and they obtained its upper and lower bounds.

The **cumulative past inaccuracy (CPI)** proposed by Kumar and Taneja [29] (2015) is defined as

$$CPI(X,Y) = -\int_0^\infty F(x)\ln G(x)dx = CPD(X,Y) + CPE(X), \tag{38}$$

where

$$CPD(X,Y) = \int_0^\infty F(x)\ln\frac{F(x)}{G(x)}dx \tag{39}$$

represents the **cumulative past relative entropy**.

The **dynamic cumulative past inaccuracy (DCPI)** proposed by Kumar and Taneja [29] (2015) (received at 15 April 2014, accepted at 12 march 2015 and was published in december 2015 in J.T.S.A.) and Kundu, Di Crescenzo and Longobardi [31] (2016) (received at 28 March 2014, accepted at 2 August 2015 and was published in 2016 in Metrika) is defined as

$$DCPI(X,Y;t) = -\int_0^t F(x;0,t)\ln G(x;0,t)dx = -\int_0^t \frac{F(x)}{F(t)}\ln\frac{G(x)}{G(t)}dx \tag{40}$$

$$= DCPD(X,Y;t) + DCPE(X;t),$$

where

$$DCPD(X,Y;t)=\int_0^t F(x;0,t)\ln\frac{F(x;0,t)}{G(x;0,t)}dx=\int_0^t \frac{F(x)}{F(t)}\ln\frac{F(x)/F(t)}{G(x)/G(t)}dx \quad (41)$$

represents the **dynamic cumulative past relative entropy**.

The **interval inaccuracy measure (II)** proposed by Kundu [30] (2015) is defined as

$$II(X,Y;t_1,t_2)=-\int_{t_1}^{t_2} f(x;t_1,t_2)\ln g(x;t_1,t_2)dx=-\int_{t_1}^{t_2}\frac{f(x)}{F(t_2)-F(t_1)}\ln\frac{g(x)}{G(t_2)-G(t_1)}dx. \quad (42)$$

The **weighted interval inaccuracy measure (WII)** proposed by Kundu [30] (2015) is defined as

$$\begin{aligned}WII(X,Y;t_1,t_2)&=-\int_{t_1}^{t_2} xf(x;t_1,t_2)\ln g(x;t_1,t_2)dx\\&=-\int_{t_1}^{t_2} x\frac{f(x)}{F(t_2)-F(t_1)}\ln\frac{g(x)}{G(t_2)-G(t_1)}dx.\end{aligned} \quad (43)$$

The **interval cumulative residual inaccuracy (ICRI)** proposed by Kundu, Di Crescenzo and Longobardi [30] (2016) is defined as

$$\begin{aligned}ICRI(X,Y;t_1,t_2)&=-\int_{t_1}^{t_2}\overline{F}(x;t_1,t_2)\ln\overline{G}(x;t_1,t_2)dx\\&=-\int_{t_1}^{t_2}\frac{\overline{F}(x)}{\overline{F}(t_1)-\overline{F}(t_2)}\ln\frac{\overline{G}(x)}{\overline{G}(t_1)-\overline{G}(t_2)}dx.\end{aligned} \quad (44)$$

The **interval cumulative past inaccuracy (ICPI)** proposed by Kundu, Di Crescenzo and Longobardi [30] (2016) is defined as

$$\begin{aligned}ICPI(X,Y;t_1,t_2)&=-\int_{t_1}^{t_2}F(x;t_1,t_2)\ln G(x;t_1,t_2)dx\\&=-\int_{t_1}^{t_2}\frac{F(x)}{F(t_2)-F(t_1)}\ln\frac{G(x)}{G(t_2)-G(t_1)}dx.\end{aligned} \quad (45)$$

CONCLUSIONS

In this paper we present a review of cumulative entropies from reliability theory. First we present the Shannon entropy concept proposed by Shannon [45] and then we present the residual and past entropy-types measures based on CDF and RF, the cumulative entropies, the cumulative relative entropies and inaccuracy measures.

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