A NEW SUZUKI TYPE FIXED POINT THEOREM

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Abstract: *In this paper we prove a fixed point result for F-Suzuki contractions.*

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1. INTRODUCTION

Banach's contraction principle (BCP) [1] is one of the initial and also fundamental results in theory of fixed point. In the literature, there are plenty of extensions of this result.

Theorem 1.1.([1]). *Let (X,d) be a complete metric space and let $T:X\to X$ a contraction ($d(Tx,Ty)\le c\cdot d(x,y),(\forall)x,y\in X, c\in[0,1)$). Then T has a unique fixed point in X.*

Several authors have obtained many extensions and generalizations of the (BCP). So, in 1962, Edelstein [2] proved the next version of contraction principle.

Theorem 1.2.([2]). *Let (X,d) be a compact metric space and let $T:X\to X$. Assume that $d(Tx,Ty)<d(x,y)$ for all $x,y\in X$ with $x\ne y$. Then T has a unique fixed point in X.*

In 2009, Suzuki [7] proved generalized versions of Edelstein's result in compact metric space as follows.

Theorem 1.3.([7]).*Let (X,d) be a compact metric space and let $T:X\to X$. Assume that*

$\left[\frac{1}{2}d(x,Tx)<d(x,y)\Rightarrow d(Tx,Ty)<d(x,y)\right]$ *for all $x,y\in X$ with $x\ne y$. Then T has a unique fixed point in X.*

Later, in 2012,Wardowski [9] generalized the Banach contraction principle in a different manner, introducing a new type of contractions called *F*-contraction.

Definition 1.4. ([9]). *Let (X,d) be a metric space. An operator $T:X\to X$ is said to be an F-contraction if there exists $\tau>0$ such that*

$$d(Tx,Ty)>0\Rightarrow\tau+F(d(Tx,Ty))\le F(d(x,y)),\ (\forall)x,y\in X \qquad (1)$$

where $F:(0,\infty)\to R$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. for all $\alpha,\beta\in(0,\infty)$, such that $\alpha<\beta, F(\alpha)<F(\beta)$;

(F2)For each sequence $\{\alpha_n\}_{n>0}$ of positive numbers $\lim\limits_{n\to\infty}\alpha_n=0$ if and only if $\lim\limits_{n\to\infty}F(\alpha_n)=-\infty$

(F3) There exists $k\in(0,1)$ such that $\lim\limits_{\alpha\to0^+}\alpha^kF(\alpha)=0$.

Theorem 1.5.([9]). *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in}$ converges to x^*.*

In 2014, Piri [5] proved the following result:

Theorem 1.6. ([5]). *Let (X,d) be a complete metric space and $T: X \to X$ be an F-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in}$ converges to x^*.*

Definition 1.7. ([5]). *Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an F-Suzuki contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \tau + F(d(Tx,Ty) \leq F(d(x,y)), \tag{2}$$

where $F: \mathrm{R}_+ \to \mathrm{R}$ is a mapping satisfying the following conditions:

(Fs1) F is strictly increasing, i.e. for all $\alpha, \beta \in (0,\infty)$, such that $\alpha < \beta, F(\alpha) < F(\beta)$;

(Fs2) $\inf F = -\infty$;

(Fs3) F is continuous on $(0,\infty)$.

In this paper, using the idea from [4] we introduced a new type of F-contraction, and will prove a fixed point theorem which generalizes some known results.

2. MAIN RESULTS

First, let F denote the family of all functions $F: \mathrm{R}_+ \to \mathrm{R}$ which satisfies the following conditions:

($F_E 1$) F is strictly increasing, that is, for all $x, y \in R_+$, if $x < y$ then $F(x) < F(y)$;

($F_E 2$) F is continuous on $(0,\infty)$.

Definition 2.1. *Let (X,d) be a complete metric space. A map $T: X \to X$ is said to be a F_E* ***-Suzuki contraction*** *on (X,d) if there exists $F \in \mathfrak{F}$ and $\tau>0$ such that for all $x, y \in X$*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow \tau + F(d(Tx,Ty)) \leq F(E(x,y)) \tag{3}$$

where

$$E(x,y) = d(x,y) + |d(x,Tx) - d(y,Ty)| \tag{4}$$

Theorem 2.2. *Let (X,d) be a complete metric space and $T: X \to X$ be an F_E-Suzuki contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in}$ converges to x^*.*

Proof: Let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_1 = Tx_0,\ x_2 = Tx_1 = T^2 x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \quad \forall n > 1 \tag{5}$$

Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathrm{N}$. Then $Tx_{n_0} = x_{n_0}$. This proves that x_{n_0} is a fixed point of T.

From now, we assume that $x_n \neq x_{n+1}$, $\forall n \in \mathrm{N}$. Then $0 < d(x_n, x_{n+1}) = d(x_n, Tx_n)$ and $\frac{1}{2} d(x_n, Tx_n) < d(x_n, Tx_n) = d(x_n, x_{n+1})$, $\forall n \in \mathrm{N}$. It follows from (3), that there exist $\tau > 0$ so that

$$\tau + F(d(Tx_n, T^2 x_n)) \leq F(E(x_n, Tx_n)) \Leftrightarrow \tau + F(d(x_{n+1}, x_{n+2})) \leq F(E(x_n, x_{n+1})) \tag{6}$$

where

$$\begin{aligned} E(x_n, x_{n+1}) &= d(x_n, x_{n+1}) + |d(x_n, Tx_n) - d(x_{n+1}, Tx_{n+1})| \\ &= d(x_n, x_{n+1}) + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})| \end{aligned}$$

If we denote by $d_n = d(x_n, x_{n+1})$ we have $E(x_n, x_{n+1}) = d_n + |d_n - d_{n+1}|$ and (6) becomes

$$\tau + F(d_{n+1}) \leq F(d_n + |d_n - d_{n+1}|) \tag{7}$$

If there exists $n \in$ such that $d_{n+1} > d_n$, then $\tau + F(d_{n+1}) \leq F(d_{n+1}) \Rightarrow \tau \leq 0.$

This is a contradiction. Then, for $d_n < d_{n+1}$, because $\tau > 0$, we have

$$\tau + F(d_{n+1}) \leq F(2d_n - d_{n+1}) \Leftrightarrow F(d_{n+1}) \leq F(2d_n - d_{n+1}) - \tau < F(2d_n - d_{n+1}) \tag{8}$$

and using ($F_E 1$), $d_{n+1} < 2d_n - d_{n+1}$, so, the sequence $\{d_n\}$ is strictly increasing and bounded.

Now, let $d = \lim_{n\to\infty} d_n$ and we suppose that $d > 0$. Because $\{d_n\} \downarrow d$ it result that $(2d_n - d_{n+1}) \downarrow d$, and taking the limit as $n \to \infty$ in (8), we get
$\tau + F(d + 0) \leq F(d + 0) \Rightarrow \tau \leq 0.$

But, this is a contradiction. Therefore,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, Tx_n) = 0\,. \tag{9}$$

In order to prove that $\{x_n\}_{n>0}$ is a Cauchy sequence in metric space (X, d), we suppose contrary, that is, there exist $\varepsilon > 0$ and the sequences $\{n(k)\}$, $\{m(k)\}$ of positiv integers with $n(k) > m(k) > k$ such that $d(x_{n(k)}, x_{m(k)}) \geq \varepsilon$ and $d(x_{n(k)-1}, x_{m(k)}) < \varepsilon$, $(\forall) k \in N$.

Then we have $\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) < d(x_{n(k)-1}, x_{n(k)}) + \varepsilon$.

Letting $k \to \infty$ and using (9) it follows that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon \tag{10}$$

From (9) and (10) it result there exist a natural number *N* such that
$\frac{1}{2} d(x_{n(k)}, x_{n(k)+1}) = \frac{1}{2} d(x_{n(k)}, T(x_{n(k)})) < \frac{\varepsilon}{2} < d(x_{n(k)}, x_{m(k)}), \quad (\forall) k \geq N.$

So, because the assumption of the theorem, we get

$$\frac{1}{2}d\left(x_{n(k)},T(x_{n(k)})\right)< d(x_{n(k)},x_{m(k)}) \Rightarrow \tau + F\left[d\left(Tx_{n(k)},Tx_{m(k)}\right)\right]\le F\left[E\left(x_{n(k)},x_{m(k)}\right)\right], \quad (\forall)k\ge N$$

$$\Leftrightarrow \tau + F\left[d\left(x_{n(k)+1},x_{m(k)+1}\right)\right]\le F\left[d\left(x_{n(k)},x_{m(k)}\right)\right.$$

Taking the limit as $k\to\infty$ and using ($F_E 2$)

$$\tau + F(\varepsilon)\le F(\varepsilon)\Rightarrow \tau\le 0.$$

It is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequences and by completeness of X there converges to some point $x^*\in X$. Therefore

$$\lim_{n\to\infty} d\left(Tx_n,x^*\right)=0. \tag{11}$$

Next, we show that x^* is a fixed point of T . For this, we claim that

$$\frac{1}{2}d\left(x_n,Tx_n\right)< d\left(x_n,x^*\right) or\ \frac{1}{2}d\left(Tx_n,T^2x_n\right)< d\left(Tx_n,x^*\right) \tag{12}$$

Assume that there exists $m\in$ N such that

$$\frac{1}{2}d\left(x_m,Tx_m\right)\ge d\left(x_m,x^*\right) and\ \frac{1}{2}d\left(Tx_m,T^2x_m\right)\ge d\left(Tx_m,x^*\right) \tag{13}$$

Then,

$$d\left(x_m,x^*\right)\le \frac{1}{2}d\left(x_m,Tx_m\right)\le \frac{1}{2}\left[d\left(x_m,x^*\right)+d\left(x^*,Tx_m\right)\right]$$

which implies that

$$d\left(x_m,x^*\right)\le d\left(x^*,Tx_m\right) \tag{14}$$

and from (13)

$$d\left(x_m,x^*\right)\le d\left(x^*,Tx_m\right)\le \frac{1}{2}d\left(Tx_m,T^2x_m\right) \tag{15}$$

Since $\frac{1}{2}d\left(x_m,Tx_m\right)< d\left(x_m,x_{m+1}\right)=d\left(x_m,Tx_m\right)$, by the assumption of theorem we get

$$F\left(d\left(Tx_m,T^2x_m\right)\right)\le F\left[E\left(x_m,Tx_m\right)\right]-\tau\le F\left[E\left(x_m,Tx_m\right)\right]$$

because $\tau>0$.

So, from ($F_E 1$) we get

$$d\left(Tx_m,T^2x_m\right)\le E\left(x_m,Tx_m\right)=d\left(x_m,Tx_m\right)+\left|d\left(x_m,x_{m+1}\right)-d\left(Tx_m,T^2x_m\right)\right|$$

$$=2d\left(x_m,Tx_m\right)-d\left(Tx_m,T^2x_m\right)\Leftrightarrow d\left(Tx_m,T^2x_m\right)\le d\left(x_m,Tx_m\right) \tag{16}$$

and from (13), (15), (16) it follows that

$$d\left(Tx_m,T^2x_m\right)< d\left(x_m,Tx_m\right)\le d\left(x_m,x^*\right)+d\left(x^*,Tx_m\right)\le d\left(Tx_m,T^2x_m\right)$$

This is a contradiction. Hence relations (12) holds.
We suppose now that $Tx^* \neq x^*$.
(1) If $\frac{1}{2}d\left(x_n,Tx_n\right)< d\left(x_n,x^*\right)$ from assumption of theorem,

$$\tau+F\left(d\left(Tx_n,Tx^*\right)\right)\le F\left(E\left(x_n,x^*\right)\right)\Leftrightarrow$$
$$\tau+F\left(d\left(x_{n+1},Tx^*\right)\right)\le F\left(d\left(x_n,x^*\right)+\left|d(x_n,x_{n+1})-d(x^*,Tx^*)\right|\right)$$

Taking the limit and using $(F_E 2)$ we have
$$\tau+F\left(d(x^*,Tx^*)\right)\le F\left(d(x^*,Tx^*)\right)\Rightarrow \tau\le 0$$

This is a contradiction.
(2) If $\frac{1}{2}d\left(Tx_n,T^2x_n\right)< d\left(Tx_n,x^*\right)$ then
$$\tau+F\left(d(T^2x_n,Tx^*)\right)\le F\left(E\left(Tx_n,x^*\right)\right)\Leftrightarrow$$
$$\tau+F\left(d(x_{n+2},Tx^*\right)\le F\left(d(x_{n+1},x^*)+\left|d(x_{n+1},x_{n+2})-d\left(x^*,Tx^*\right)\right|\right)$$

So, taking the limit when:
$$\tau+F\left(x^*,Tx^*\right)\le F\left(x^*,Tx^*\right)\Rightarrow \tau\le 0$$

Hence x^* is a fixed point of T.

Finally, we prove that the fixed point of T is unique. For this, let x^*, y^* be two fixed points of T and suppose that $Tx^* = x^* \neq y^* = Ty^*$, so $d(x^*,y^*)>0$.

Because $E\left(x^*,y^*\right)=d(x^*,y^*)+\left|d(x^*,Tx^*)-d\left(y^*,Ty^*\right)\right|=d(x^*,y^*)$ it follows that

$$0=\frac{1}{2}d(x^*,Tx^*)< d\left(x^*,y^*\right)\Rightarrow \tau+F\left(d\left(Tx^*,Ty^*\right)\right)\le F\left(E\left(x^*,y^*\right)\right)\Leftrightarrow$$
$$\Leftrightarrow \tau+F\left(d\left(x^*,y^*\right)\right)\le F\left(E\left(x^*,y^*\right)\right)\Rightarrow \tau\le 0.$$

It is a contradiction. Then, $d(x^*,y^*)=0$, that is $x^*=y^*$. This proves that the fixed point of T is unique.

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