A COMPARISON BETWEEN DETRENDING METHODS: HODRICK-PRESCOTT, BAXTER-KING, CHRISTIANO-FITZGERALD FILTERS. A SHORT SURVEY

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Abstract: *This article examines the differences between the extracted cyclical components of some macro-economic time series using three detrending methods: HP (Hodrick-Prescott), BK (Baxter-King) and CF (Christiano-Fitzgerald). We use different approaches to compare the differences. A standard examination of the cyclical component is done. We also use a frequency domain approach and examine the sample spectra for each cycle. Furthermore, impulse responses and the correlation between the cyclical components extracted by each detrending method are studied. Our conclusion is that for quarterly data HP, BK and CF produce similar cycles. However, when we considered annual data the HP method gives us significant differences from the BK and CF methods*.

Keywords: *Hodrick-Prescott filter, BK (Baxter-King) filter, CF (Christiano-Fitzgerald) filter, DSP (Digital Signal Processing), macro-economy.*

1. INTRODUCTION

Correctly estimating business cycles is important for macro-economic research. Several methods of extracting business cycles from some given time series were developed, but none of these lead to different results one to another. This is the reason why the authors from [22], investigate these three de-trending tools: HP (Hodrick-Prescott), BK (Baxter-King) and CF (Christiano-Fitzgerald). All these three are also considered to be filters, because their aim is to separate the trend component from the cyclical component. The analyses carried out in this article include a basic analysis of the cyclical component, a frequency domain examination of the spectral densities of each filter, also the correlation between the cyclical components found by the detrending methods and the impulse responses are also studied.

Also, the purpose of this research article is to determine, in future researches, what other types of signals, besides the time series presented here, would be suitable to adapt these filters (HP, BK or CF) to extract important observations from input data. By adapting these types of filters to other datasets we mean to modify not only the constants involved in their own definition, but also the mathematical definitions or equations that describe them.

From the conclusions obtained in [22], we are now sure that for one quarter of the data, the HP, BK and CF filters extract similar cycles and this is an very important result.

The second important result that could be used for other datasets concerns the behavior of the HP filter for annual data, which differs from the other two band pass filters (BK and CF).

2. MAIN RESULTS CONCERNING THE HODRICK-PRESOTT FILTER, THE BAXTER-KING FILTER, THE CHRISTIANO-FITZGERALD FILTER

2.1 The Hodrick-Prescott Filter

For this section we follow the line from [21].

The Hodrick-Prescott (HP) filter is a standard tool in macroeconomics for differentiating the long trend in a data series from short run fluctuations. It is also a smoothing method whose aim is to obtain a smooth component from the trend. Let's assume we have the following time series:

$y_t = \tau_t + c_t, t = 1,2, \ldots, T.$

The HP filter smoothed the series $\hat{\tau}_T = (\hat{\tau}_{T1}, \hat{\tau}_{T2}, \ldots, \hat{\tau}_{TT})$ as defined and described in economics by Hodrick and Prescott (1980, 1997) ([3] and [4]) results from minimizing, over all $\tau \in \mathbb{R}^T$,

$$\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}(\tau_{t+1} - 2\tau_t + \tau_{t-1})^2, \qquad (1)$$

where T denotes the sample size, λ is the nonnegative smoothing parameter so that for quarterly of the data is often chosen to be equal to 1600, and $y = (y_1, y_2, \ldots, y_T)'$ is the data series to be smoothed.

In [5] a similar filtering technique was introduced [1]. Usually, $\hat{\tau}_{Tt}$ are referred to as "trend components", while $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt}$ it is named "cyclical component".

It was mentioned in [1] that there exists a minimizer, which is unique, to the minimization problem for equation (1), so that, for a known positive definite $(T \times T)$ matrix F_T, letting I_T denote the $(T \times T)$ identity matrix, $y = (\lambda F_T + I_T)\hat{\tau}_{Tt}$ and

$\hat{\tau}_{Tt} = (\lambda F_T + I_T)^{-1} y;$ (2).

So the trend component $\hat{\tau}_{Tt}$ and the cyclical component $\hat{c}_{Tt}$ are both weighted averages of y_t, so therefore we will re-write:

$$\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts}\, y_s\,.$$

For notational convenience, the dependence of w_{Tts} and $\hat{\tau}_{Tt}$ on λ is suppressed. One of the purposes set in [1] is to find a new representation for w_{Tts} and see immediate consequences of this representation. This approach eliminates the inability to discover a simple expression for the elements of $(\lambda F_T + I_T)^{-1}$, which prevented other researchers from finding a simple expression for the weights that are implicit for the HP filter (for more, see [1]).

We notice first of all, that: $\hat{\tau}_{Tt}(y_1 + 1, y_2 + 1, \ldots, y_T + 1) = \hat{\tau}_{Tt}(y_1, y_2, \ldots, y_T) + 1$, so this means that $\sum_{s=1}^{T} w_{Tts} = 1$ for $t \in \{1,2, \ldots, T\}$. Also, we have that: $\hat{\tau}_{Tt}(1,2, \ldots, T) = t$, and therefore we have that: $\sum_{s=1}^{T} w_{Tts} s = t$, for $t \in \{1,2, \ldots, T\}$.

Authors in [1] obtained that this way, a quadratic trend is not absorbed in $\hat{\tau}_{Tt}$.

They also mentioned that previous literature on the HP filter is only based on the observation that the first order condition for $\hat{\tau}_{Tt}, t \in \{3, \ldots T - 2\}$ is:

$$-2(y_t - \hat{\tau}_{Tt}) - 4\lambda\left(\hat{\tau}_{T,t+1} - 2\hat{\tau}_{Tt} + \hat{\tau}_{T,t-1}\right) + 2\lambda\left(\hat{\tau}_{Tt} - 2\hat{\tau}_{T,t-1} + \hat{\tau}_{T,t-2}\right) + 2\lambda\left(\hat{\tau}_{T,t+2} - 2\hat{\tau}_{T,t+1} + \hat{\tau}_{Tt}\right) = 0 \qquad (3)$$

Let $\bar{B}$denote the forward operator and $B$the backward operator, then according to [1], this simplifies to the following relation:

$$y_t = (\lambda\bar{B}^2 - 4\lambda\bar{B} + (1+6\lambda) - 4\lambda B + \lambda B^2)\hat{\tau}_{Tt}, \quad (4)$$

which can also be re-written as:

$$y_t = (\lambda|1-B|^4 + 1)\hat{\tau}_{Tt} \quad (5)$$

Papers that, according to [1], that analyze the HP filter based on the first order condition are for example [6], [7], [8], [9] and [10].

A high value of λ will give a more linear trend and will allow for increased variation in the cyclical component, [22]. For $\lambda = 0$, the trend component is, obviously, equivalent to the actual time series, y_t.

The determinant key for the minimization problem (1) is the value of λ.

2.2 The Baxter-King filter

For following we consider the results from [22].

BK filter is decomposing a time series, let's say y_t, into three different components: trend, cycle and the irregular component.

$$y_t = \tau_t + c_t + \varepsilon_t \quad (6)$$

where τ_t is the trend component, c_t is the cyclical component, ε_t is the irregular component, [15].

As a result, a new time series $y_t{}^*$ is obtained when we apply a finite symmetric moving average.

So, we define the following symmetric moving average: $y_t{}^* = \sum_{k=-K}^{K}\beta_k\, y_{t-k}$, (7) where β_k are fixed constants or let's say weights and K the maximum lag length. In order to extract the cyclical components from the above time series, BK uses weights that will add up to zero, meaning: $\sum_{k=-K}^{K}\beta_k = 0$.That will be the trend elimination property, [15]. These weights that add up to zero and the moving average have some good elimination properties that generate a so called stationary time series.

This is an very important fact, because economic time series have the tendency to be non-stationary, [15].

Authors in [15] derive this filter through the frequency-domain perspective. The point from where to start with the BK filter is, according to [22], the Cramer representation theorem, which states the following:

$$y_t = \int_{-\pi}^{\pi}\xi(\omega)d\omega \quad (8)$$

under some suitable conditions. In this representation, the time series is now written as the integral of the random periodic components, $\xi(\omega)$, and where ω is expressed in radians. According to [22], if we apply Cramer Theorem to equation (7), we have the following:

$$y_t{}^* = \int_{-\pi}^{\pi}\beta(\omega)\xi(\omega)d\omega \quad (9)$$

where $\beta(\omega) = \sum_{k=-K}^{K}\beta_k \exp{(i\omega k)}$ it is the frequency response function for this filter.

Another interpretation, according to [22] would be: how much ${y_t}^*$ responds to y_t at a given frequency ω with respect to the weight $\beta(\omega)$, also to the random and periodic component $\xi(\omega)$. An important observation for the BK filter would be that $\beta(\omega)$ has the value 0 at frequency 0, meaning: $\beta(0) = 0$.

The cyclical component is defined or extracted as follows:

$$c_t = \sum_{j=-K}^{K} b_j \, y_{t-j} \tag{10}$$

where the weights b_j can found by applying the inverse Fourier transform to the frequency response function, [12].

$$b_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \beta(\omega) \exp(i\omega j) d\omega \tag{11}$$

Considering the definition from [22], ω has to be between $\frac{\pi}{16}$ and $\frac{\pi}{3}$.The construction of BK filter is made from two low pass filters and is having two frequency bands, $\frac{\pi}{16}$ and $\frac{\pi}{3}$.Because of this, we define two frequency response functions: $\beta_1(\omega) = 1$, for $|\omega| \leq \omega_1 = \frac{\pi}{16}$ (zero otherwise), $\beta_2(\omega) = 1$, for $|\omega| \leq \omega_2 = \frac{\pi}{3}$. To obtain the weights b_j, we get $\beta_1(\omega)$ from $\beta_2(\omega)$ and we will have the desired frequency response function for $\frac{\pi}{16} \leq \omega \leq \frac{\pi}{3}$ or $-\frac{\pi}{3} \leq \omega \leq -\frac{\pi}{16}$ and zero otherwise.The weights b_j can then be obtained from equation (11).

2.3 The Christiano and Fitzgerald filter

The Christiano and Fitzgerald filter (CF), like the BK filter, is the approximation of an ideal band pass filter. Assuming a symmetric moving average is not the case here [16].The cyclical component, according to [22], is given by:

$$c_t = b_0 y_t + b_1 y_{t+1} + \cdots + b_{T-1-t} y_{T-1} + \tilde{b}_{T-t} y_t + b_1 y_{t-1} + \cdots + \tilde{b}_{T-1} y_1 + b_{t-2} y_2 \tag{12}$$

For t=3,4,…,T-2, where:

$$b_j = \frac{\sin(jc) - \sin(ja)}{\pi j}, j \geq 1$$

$$b_0 = \frac{c-a}{\pi}, a = \frac{2\pi}{p_h}, c = \frac{2\pi}{p_h}$$

$$\tilde{b}_k = -\frac{1}{2} b_0 - \sum_{j=1}^{k-1} b_j$$

and p_t and p_h are defined like in [16].So, cycles that are longer than p_t but shorter than p_h are defined to be the actual cyclical component c_t. The CF filter is not symmetric, [17].

This is in contrast with BK filter, which is considered to be symmetric. CF is consistent, when compared to BK filter, because it converges to an ideal band pass filter when the sample size T is increased, [22], [17].

3. THE IMPULSE RESPONSE FUNCTION

The Impulse Response Function (IRF) is in general obtained from Variance Autoregressive model (VAR). Consider the following VAR (1) systems with these two equations:

$$x_t = a_1 + b^1{}_{1,1}x_{t-1} + b^1{}_{1,1}x_{t-1} + b^1{}_{1,2}z_{t-1} + \varepsilon_{t,x}$$

$$z_t = a_2 + b^1{}_{2,1}x_{t-1} + b^1{}_{1,1}x_{t-1} + b^1{}_{2,2}z_{t-1} + \varepsilon_{t,z}$$

where x_t and z_t appear as is [22].This shows that if there is a shock in x_t, there will also be an effect in z_t.

Let's suppose that at a given time t there is a shock to z by one standard deviation, σ_z , and x is not shocked. Furthermore, suppose that for period: $t+1, t+2, \dots t+n$ we don't have any shock in either x or z [20]. This response to the shock will then be:

- Time t (when the shock occurs):
  The effect on z is σ_z and on x we have no effect.
- Time $t+1$:
  The effect on z is $\beta^1{}_{2,2}\sigma_z$ and the effect on x is $\beta^1{}_{1,2}\sigma_z$
- Time $t+2$:
  The effect on z is $(\beta^1{}_{2,2})^2\sigma_z + \beta^1{}_{2,1}\beta^1{}_{1,2}\sigma_z$ and the effect on z is $\beta^1{}_{1,2}\beta^1{}_{2,2}\sigma_z + \beta^1{}_{1,1}\beta^1{}_{1,2}\sigma_z$

In future periods, $t+n$, the effect from the shock will be different from the values of $\beta^1{}_{i,j}$.

The coefficients $\beta^k{}_{i,j}$ are obtained from the following VAR(k) system, with two equations, [22]:

$$\tau_t = \alpha_1 + \beta^1{}_{1,1}\tau_{t-1} + \beta^1{}_{1,2}c_{t-1} + \cdots + \beta^k{}_{1,1}\tau_{t-k} + \beta^k{}_{1,2}c_{t-k} + \varepsilon_{1,t},$$

$$c_t = \alpha_2 + \beta^1{}_{2,1}\tau_{t-1} + \beta^1{}_{2,2}c_{t-1} + \cdots + \beta^k{}_{2,1}\tau_{t-k} + \beta^k{}_{2,2}c_{t-k} + \varepsilon_{2,t},$$

where: α_i is a constant $(i = 1,2)$, τ_t is obviously the original time series and c_t is the cyclical component for the time series. Above, k represents the number of lags and $\varepsilon_{i,t}$ is a stochastic error term $(i = 1,2)$.

In order to decide about the number of lags in this the model, we proceeded as in [22].

4. CONCLUSIONS

This survey article reveals the new mathematically rigorous results and properties of the HP filter, BK filter, CF filter obtained by the authors in [22] and also establishes new lines of research in this field, having set for the future clear objectives concerning the kind of results that are to be obtained.

Future work will include possible applications of these filters in DSP, by calibrating the constants in each filter after replacing the GDP, consumption, investment and inflation test datasets with digitized samples from complex valued signals, reconstruct the impulse response functions for each filter and after plot the spectral density for each filter in the detrended dataset.

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