REPRESENTING FUZZY SYSTEMS UNIVERSAL APPROXIMATORS

Iuliana IATAN[*], Stefan GIEBEL[**]

[*]Technical University of Civil Engineering, Bucharest, Romania (iuliafi@yahoo.com)
[**]Technical University of Braunschweig, Germany

DOI: 10.19062/1842-9238.2018.16.2.5

Abstract: *The described representation of a fuzzy system enables an approximate functional characterization of the inferred output of the fuzzy system. With polynomial subsystem inferences, the approximating function is a sum of polynomial terms of orders depending on the numbers of input membership functions. The constant, linear, and nonlinear parts of the fuzzy inference can hence be identified. The present work also includes two applications which show that the procedure can very well approximate the differential equations. In the case of no analytical solution, the procedure is a good alternative.*

Keywords: *fuzzy systems, approximating function, universal approximators, inferred output*

1. INTRODUCTION

The aim of the paper is to prove that fuzzy systems are also universal approximators to continuous functions on compact domain in the case of the described subsystem inference representation corresponding to the fuzzy systems, as in the work of Kosko [11], Wang [16] and later Alci [1], Kim [10]. The paper is organized as follows: the first section provides a brief review on product sum fuzzy inference and introduces the concepts of additive and multiplicative decomposable systems; the second section presents a subsystem inference representation; the next sections discuss the cases of polynomial, sinusoidal, orthonormal and other designs of subsystem inferences; the last section presents some conclusions on the matter.

2. A FUZZY SYSTEM WITH TWO INPUT VARIABLES

A fuzzy system of *n* input variables $a, b, \ldots, y, z,$ with input membership functions $A_i, i=\overline{1,m_a},\ B_j,\ j=\overline{1,m_b}, \ldots, Y_h, h=\overline{1,m_h}, Z_r,\ r=\overline{1,m_r}$ is expressible as an additive sum of $m_a \times m_b \times \ldots \times m_y \times m_z$ systems, each of which is multiplicative, and thus decomposable into *n* single variable subsystems.

Consider a fuzzy system with two input variables *a* and *b* with rule consequents embedded in the $m_a \times m_b$ matrix U_{m_a,m_b} from [7,8,15].

The inferred output is [7,8,15]:

$$\Im_{a,b}\left(U_{m_a,m_b}\right)=\sum_{q=1}^{m_a}\sum_{l=1}^{m_b}\gamma_{q,l}\cdot\Im_a\left(f_A^q\right)\cdot\Im_b\left(f_B^l\right), \tag{1}$$

where:

- $\gamma_{q,l}$ are the elements of the matrix ν_{m_a,m_b}, defined in [7,8,15];
- $f_A^q = \left(f_A^q(1) \quad f_A^q(2) \quad \ldots \quad f_A^q(m_a)\right)$, $q = \overline{1, m_a}$ means a set of linear independent m_a by one column vectors, selected for variable *a* and is associated to a subsystem $A^{(q)}$;
- $\Im_a\left(f_A^q\right)$ represents the inferred output of subsystem $A^{(q)}$;
- $f_B^l = \left(f_B^l(1) \quad f_B^l(2) \quad \ldots \quad f_B^l(m_b)\right)$, $l = \overline{1, m_b}$ means a set of linear independent m_b by one column vectors, selected for variable *b*;
- $\Im_b\left(f_B^l\right)$ represents the inferred output of subsystem $B^{(l)}$.

The selection of the vectors f_A^q and f_B^l should depend on the kind of approximation function one desires to use for the problem at hand, be it polynomial, sinusoidal, or other designs.

3. POLYNOMIAL SUBSYSTEM INFERENCE

The vectors $f_A^q, q = \overline{1, m_a}$ can be selected to emulate polynomial functions (they are termed polynomial subsystem vectors). The resulting subsystem inference $\Im_a\left(f_A^q\right)$ represents the polynomial subsystem inferences. In the case of a system with *n* fuzzy variables $a, b, \ldots, y, z,$ having $m_a, m_b, \ldots, m_y, m_z$ input membership functions, the inferred output is an approximation to the polynomial function, which contains polynomial terms up to orders of $m_a - 1, m_b - 1, \ldots, m_y - 1$ and $m_z - 1$ in $a, b, \ldots, y$ and *z*. Conversely, the polynomial function can be considered as an approximate output of the fuzzy system. The subsystem inference representation contributes to an approximate functional characterization of the inferred output in the sense that as $m_a, m_b, \ldots, m_y$ and m_z tend to large values, the polynomial inferences $\Im_a\left(f_A^i\right), \Im_b\left(f_B^j\right), \ldots, \Im_y\left(f_Y^h\right)$ and $\Im_z\left(f_Z^r\right)$ converge uniformly to the polynomial terms $a^{i-1}, b^{j-1}, \ldots, y^{h-1}$ and z^{r-1}.

4. SINUSOIDAL AND EXPONENTIAL SUBSYSTEM INFERENCES

Same as before for polynomial inferences, in the example of sinusoidal subsystem inferences, the fuzzy inferred output constitutes a piecewise linear approximation of a sinusoidal/ cosinusoidal function. Using the sinusoidal inferences, the approximating function is comprised of sine/ cosine, and cross product terms. With appropriate designs [7,8,15], sinusoidal inferences can be further manipulated into an orthonormal set.

In the example of exponential subsystem inferences, the inference of the fuzzy system constitutes [7,8,15] a piecewise linear approximation to an exponential term.

5. APPLICATIONS

In the case of the first application, we shall use the polynomial and exponential inferences together for the fuzzy approximation of the differential equation solution

$$\begin{cases} y''' = x\,\mathrm{e}^{-x} \\ y(0) = 0 \\ y'(0) = 2 \\ y''(0) = 2 \end{cases} \tag{2}$$

We shall consider the fuzzy systems with two fuzzy variables: *a* is for approximating the polynomial term of *x*, and variables *b* is for realizing the exponential term e^{-x}.

Trapezoidal input membership functions in Fig. 1 and Fig. 2 are assumed [7] for all variables.

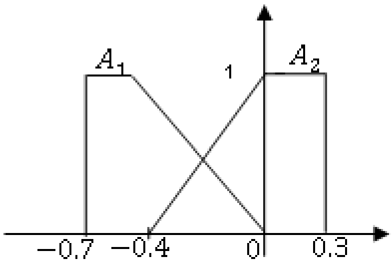

FIG. 1. Trapezoidal input membership functions for the fuzzy variable *a*

Let the domains of interest for *a* and *b* be [-0.7; 0.3] and [0.2; 0.8].

From the Fig. 1 we can notice that:

$$\mu_{A_1}(a) = \begin{cases} 1, & a \in [-0.7, -0.4] \\ 1 - \dfrac{a+0.4}{0.4}, & a \in (-0.4, 0] \end{cases} \tag{3}$$

and

$$\mu_{A_2}(a) = \begin{cases} \dfrac{a+0.4}{0.4}, & a \in [-0.4, 0] \\ 1, & a \in (0, 0.3]. \end{cases} \tag{4}$$

For the variables *a* and *b*, one sets:

$$\begin{cases} m_a = m_b = 2 \\ a_1 = -0.4,\ a_2 = 0 \\ b_1 = 0.3,\ b_2 = 0.7 \\ f_A^{(1)} = f_B^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ f_A^{(2)} = \begin{pmatrix} 0 \\ a_2 - a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.4 \end{pmatrix} \\ f_B^{(2)} = \begin{pmatrix} 1 \\ \mathrm{e}^{b_1 - b_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{e}^{-0.4} \end{pmatrix}. \end{cases} \tag{5}$$

The inferred output of subsystem $A^{(2)}$ will be [7,8,15]:

$$\Im_a\left(f_A^2\right)= f_A^2(1)\cdot \mu_{A_1}(a)+ f_A^2(2)\cdot \mu_{A_2}(a), \tag{6}$$

namely

$$\Im_a\left(f_A^2\right)=\begin{cases} 0.4\cdot\dfrac{a+0.4}{0.4}, & a\in[-0.4,0] \\ 0.4, & a\in(0,0.3]. \end{cases} \tag{7}$$

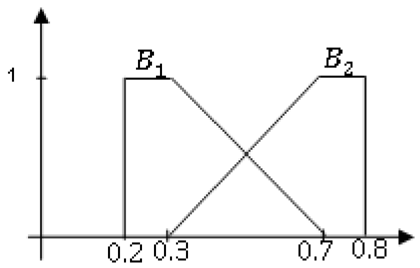

FIG. 2. Trapezoidal input membership functions for the fuzzy variable *b*

By observing Fig. 2, we can notice that:

$$\mu_{B_1}(b)=\begin{cases} 1, & b\in[0.2, 0.3) \\ 1-\dfrac{b-0.3}{0.4}, & b\in[0.3, 0.7] \end{cases} \tag{8}$$

and

$$\mu_{B_2}(b)=\begin{cases} \dfrac{b-0.3}{0.4}, & b\in[0.3, 0.7] \\ 1, & b\in(0.7,0.8]. \end{cases} \tag{9}$$

The inferred output of subsystem $B^{(2)}$ is [7]:

$$\Im_b\left(f_B^2\right)= f_B^2(1)\cdot \mu_{B_1}(b)+ f_B^2(2)\cdot \mu_{B_2}(b), \tag{10}$$

namely

$$\Im_b\left(f_B^2\right)=\begin{cases} 1, & b\in[0.2, 0.3) \\ 1+\left(1-\mathrm{e}^{-0.4}\right)\cdot\dfrac{b-0.3}{0.4}, & b\in[0.3, 0.7] \\ \mathrm{e}^{-0.4}, & b\in(0.7,\ 0.8]. \end{cases} \tag{11}$$

$$\Im_b\left(f_B^2\right)=\begin{cases} e^{-0.4}, & x\in[0.2231, 0.3567) \\ 1+\left(1-e^{-0.4}\right)\cdot\dfrac{x-0.3}{0.4}, & x\in[0.3567, 1.204] \\ 1, & x\in(1.204, 1.6092]. \end{cases} \tag{12}$$

As

$$\Im_{a,b}\left(U_{2,2}\right)=\sum_{q=1}^{2}\sum_{l=1}^{2}\nu_{q,l}\cdot\Im_a\left(f_A^q\right)\cdot\Im_b\left(f_B^l\right)=\nu_{2,2}\cdot\Im_a\left(f_A^2\right)\cdot\Im_b\left(f_B^2\right), \tag{13}$$

a fuzzy system to achieve (2) is [7]:

$$y'''=\Im_{a,b}\left(U_{2,2}\right)=\begin{cases} 0.4\cdot e^{-0.4}, & x\in[0.2231, 0.3] \\ 0, & x\in[0.4, 0]\cup(0.3, 1.6092]. \end{cases} \tag{14}$$

where:

$$\begin{cases} y'(0)=2 \\ y''(0)=2 \end{cases}$$

and $U_{2,2}$ is given by [7].

Equation (14) will be analytically solved in Matlab 7.0, using the function **dsolve**; let $y_1(x)$ be the analytical solution of the equation (2), $y_2(x)$ and $y_3(x)$ the analytical solutions of the equation (14) for $x\in[0.4, 0]\cup(0.3, 1.6092]$ and respectively for $x\in[0.2231, 0.3]$.

We shall obtain:

$$|y_1(-0.1)-y_2(-0.1)|=4.3376\text{e-006}$$
$$|y_1(0.4)-y_2(0.4)|=9.118\text{e-004}$$
$$|y_1(0.7)-y_2(0.7)|=0.0076$$
$$|y_1(1.25)-y_2(1.25)|=0.0636$$
$$|y_1(0.25)-y_2(0.25)|=5.5080\text{e-004}$$
$$|y_1(0.3)-y_2(0.3)|=9.0670\text{e-004}.$$

Especially in the case of having no analytical solution at hand, the new procedure is interesting. The Lotka-Volterra equations are also called the predator-prey equations. The equations are a pair of first-order, non-linear, differential equations. They are needed to describe [8] the dynamics of biological systems in which two species interact with each other.

One is the predator and the other, its prey. If there are not enough preys, the population of predators will decrease. And if the population of preys increases, the predator population will also increase.

Furthermore, the Lotka-Volterra equations [6] are used in economics. Similar relations are established between different kinds of industries, as an example between engine construction and mining. Furthermore, the economic cycle in general can be simulated.

They develop in time according to the pair of equations [8]:

$$\begin{cases} \dfrac{\mathrm{d}\,x}{\mathrm{d}\,t} = x(\alpha - \beta y) \\ \dfrac{\mathrm{d}\,y}{\mathrm{d}\,t} = -y(\gamma - \delta x) \end{cases} \tag{15}$$

where:

- $y = 10$ is the number of predators (for example, lions);
- $x = 800$ is the number of preys (for example, zebras);
- $\dfrac{\mathrm{d}\,x}{\mathrm{d}\,t}$ and $\dfrac{\mathrm{d}\,y}{\mathrm{d}\,t}$ represent the growth of the two populations against time;
- t represents the time;
- $\alpha = 3$, $\beta = 0.1$, $\gamma = 0.8$ and $\delta = 0.002$ are parameters representing the interaction of the two species.

The development of each species during a certain time interval can also be described by the upper procedure in the form of a polynomial. The values for *t* should be adapted to the procedure.

Fig. 3 shows that the method is useful if we seek to approximate (green) the function of one population(red).

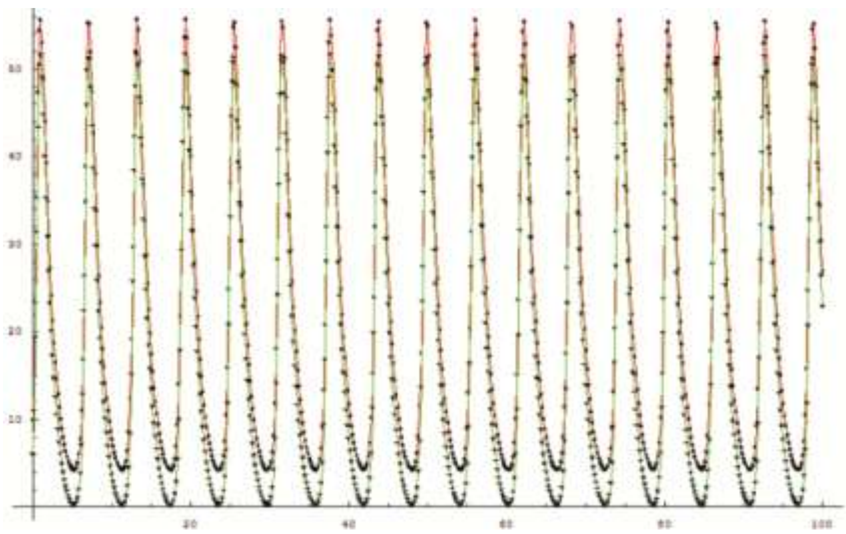

FIG. 3. The approximative solution of the population

The number of preys can be approximated easily by using the procedure above.

CONCLUSIONS

A new representation for fuzzy systems in terms of additive and multiplicative subsystem inferences of single variables is presented to prove that fuzzy systems are universal approximators to continuous functions on compact domain.

This representation enables an approximate functional characterization of the inferred output. The form of the approximating function depends on the choice of polynomial, sinusoidal, or other designs of subsystem inferences.

With polynomial subsystem inferences, the approximating function is a sum of polynomial terms of orders depending on the numbers of input membership functions.

Since polynomials are universal approximators [7,8,15], the same can be concluded regarding fuzzy systems.

With proper scaling, the sinusoidal inferences produce a set of orthonormal inferences.

The present work also includes two applications about constructing a fuzzy approximator for a function expressible in terms of sums and products of functions of a single variable. In this case, subsystem inferences that emulate the various single variable functions are adopted.

The second application [8] shows that the presented procedure can very well approximate a differential equation. In the case of no analytical solution the procedure, this is a good alternative.

REFERENCES

[1] M. Alci, Fuzzy rule-base driven orthogonal approximation, *Neural Computing & Applications*, vol. 17, no. 5-6, pp. 501-507, 2008;

[2] Y. Chen, B. Yang, A. Abraham and L. Peng, Automatic Design of Hierarchical Takagi-Sugeno Type Fuzzy Systems Using Evolutionary Algorithms, *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 3, pp. 385-397, 2007;

[3] M.T. Gan, M. Hanmandlu and A. H. Tan, From a Gaussian Mixture Model to Additive Fuzzy Systems, *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 3, pp. 303-315, 2005;

[4] G. Gandolfo. Giuseppe Palomba and the Lotka-Volterra equations. *Rendiconti Lincei*, vol. 19, no. 4, pp. 347-357, 2008;

[5] U. Grenander and M. Miller, *Pattern Theory from Representation to Inference,* Oxford University Press, New York, 2007;

[6] L. Hongxing, *From Probability Representations of Fuzzy Systems*, Science China Information Sciences, vol. 49, no. 3, pp. 339-363, 2006;

[7] I. Iatan, *A New Representation for The Fuzzy Systems in Terms of Some Additive and Multiplicative Subsystem Inferences,* Journal of Computational Analysis and Applications, vol. 14, no. 1, pp. 143-164, 2012;

[8] I. Iatan and S. Giebel, *A New Representation for The Fuzzy Systems*, in B. Hu et al. (eds.), *Operations Research Proceedings*, pp. 207-212, Selected Papers of the Annual International Conference of the German Operations Research Society (GOR) at Universität der Bundeswehr München, September 1–3, 2010, Springer, 2010;

[9] Y. Jin, Advanced Fuzzy Systems Design and Applications, Physica-Verlag Heidelberg, New York, 2003;

[10] J. Kim and J. S. Lee, *Single-Input Single-Output Convex Fuzzy Systems as Universal Approximators for Single-Input Single-Output Convex Functions*, pp. 350-355, Proceedings of the 18th International Conference on Fuzzy Systems, 2009;

[11] B. Kosko, Fuzzy Systems as Universal Approximators, *IEEE Transactions on Computers*, vol. 43, no. 11, pp. 1329-1333, 1994;

[12] Y. Liu, G. Chen and M. Ying, *Fuzzy Logic, Soft Computing and Computational Intelligence,* vol. 1, Tsinghua University Press and Springer, Beijing, 2005;

[13] Z. Miao, X. You and H. Zhen, A New Method for Fuzzy System Representation and Implementation, *Fuzzy Systems and Knowledge Discovery*, vol. 1, pp. 142-146, 2008;

[14] Y. Shi, Fuzzy Inference Modeling Based on Fuzzy Singleton-Type Reasoning, *International Journal of Innovative Computing, Information and Control*, vol. 3, no. 1, pp. 13-20, 2007;
[15] Y. Yam, Subsystem Inference Representation for Fuzzy Systems Based Upon Product-Sum-Gravity Rule, *IEEE Transactions on Fuzzy Systems*, vol. 5, no. 1, pp. 90-107, 1997;
[16] L. X. Wang, J. M. Mendel L. X. Wang and J. M. Mendel, Fuzzy Basis Functions, Universal Approximation, and Orthogonal Least Square Learning, *IEEE Trans. Neural Networks*, vol. 3, pp. 807-814, 1992.