HANDLING DYNAMIC NONLINEARITIES IN UAV AUTOMATIC FLIGHT CONTROL SYSTEMS

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DOI: 10.19062/1842-9238.2022.20.1.5

Abstract: *The linear or the nonlinear feature of systems being modelled or designed is a core property we should be familiar with before taking up any system design or analysis-related task. The dynamic model-based closed loop control design is widely used in control engineering when dynamic or static nonlinearities are considered to be present in the closed loop control system. This paper addresses both dynamical and static nonlinearities modelled and handled in the closed loop control system.*

Keywords: *nonlinearity, static nonlinearity, dynamical nonlinearity, nonlinear systems, nonlinear system analysis and modelling.*

1. INTRODUCTION

The nonlinear feature of control systems is a crucial issue in control engineering when to handle nonlinearities and select design methods for closed loop control system. It is well-known from mathematics that the ideal linear function of the form of $y = f(x) = ax + b$ rarely can appear as a pure linear function; moreover, represents a simple function from among those existing and mathematical analysis.

The linearity, and the strong wish to linearize in the nonlinear world at any price, at any loss of originality of the system dynamics is known from many centuries and decades. Taylor B. in 1715 introduced his method called (first order) Taylor-series expansion to linearize any nonlinear (polynomials, power functions, logarithm function, exponential function, trigonometric functions, (square) root, geometric series, binomial series, hyperbolic functions etc.) function. This method often called as tangent linearization of the given nonlinear function at the given equilibrium point set previously.

In flight dynamics several nonlinear functions (e.g. basic formulas of the International Standard Atmosphere, like $T(h)$, $p(h)$, or $\rho(h)$) are used to express different nonlinear relationships between physical values. It is well-known from flight mechanics, that both longitudinal and lateral/directional motion equations involve different coefficients of forces and moments being in nonlinear dependency with speed, Reynold's number etc. Moreover, many coefficients (e.g. $C_L(\alpha)$, $C_D(v)$ etc.) are nonlinear functions of several independents.

Thus, any aircraft model, among those of the UAV's models are derived as a nonlinear being linearized at some equilibrium flight regime we consider for the linearization.

Besides dynamic nonlinearities being handled in any closed loop control design there are several types of static nonlinearities should be considered and handled.

Static nonlinearities (e.g. dead zone (delay), saturation, hysteresis, relay etc.) are integral parts of devices used to build up closed loop control systems.

Existence of the static nonlinearities might be unavoidable in closed loop control systems, like it happens in case of hydraulic actuator remotely controlled by electro-magnetic valve having dead zone in control rod position, and having saturation in the cylinder of the actuator.

Next example when the dead-zone plus saturation static nonlinearity is used is 2D electro-mechanical gyroscope supplemented with potentiometer to harvest electrical signal of the rotational speed. The potentiometer itself often has a dead zone to eliminated unwanted signals from sources other than the rotational motion, like high frequency vibrations. The maximum of the rate (saturation) is defined by the aircraft flight characteristics.

This paper will address both static and dynamical nonlinearities of the small UAV automatic flight control systems, and will highlight methods serving nonlinear control system analysis and design purposes.

2. RELATED WORKS AND PRELIMINARIES

The early and pioneer work of A. Isidori (editions of 1985, 1989, and 1995) gives thorough and rigorous mathematical background of both SISO and MIMO nonlinear feedback systems dealing with system analysis and closed loop system design [5].

The linear and nonlinear systems are discussed in [2, 3, 4, 12, 13]. In [15] and [16] linearization of the nonlinear systems is presented.

In [14] a human-in-loop problem is subjected to thorough analysis, how the LTI system output nonlinearity effects the entire system performance.

The describing function method and its inversion serves nonlinear system analysis since many decades. References of [6, 7, 8, 9, 10], and [11] deal with nonlinear control system analysis using describing functions.

In [17] and [18] gives sufficient help in computer-aided design and analysis of nonlinear systems using MATLAB® and Simulink®. In [19] mathematical backgrounds of the topic addressed in this paper are discussed going into deep details.

Finally, [1] derives nonlinear dynamical aircraft models used in design and analysis of the automatic flight control systems.

3. NONLINEARITIES OF UAV AUTOMATIC FLIGHT CONTROL SYSTEMS

The automatic flight control systems have several different nonlinearities being static or dynamic. Static nonlinearities such as the dead zone, saturation, relay, inverse relay, hysteresis etc. are integral parts of any device (e.g. amplifiers, valves, actuators, induction motors, gears etc.).

Dynamic nonlinearities represent functions differing from the unique linear function. They might have different mathematical forms like trigonometric functions and their inverses, logarithm functions, exponential functions, powers of different orders excluding the first order, different types of the root function etc.

3.1 Dynamic nonlinearities of closed loop automatic flight control systems. There are several powerful linearization techniques well-known from mathematics, and widely applied and used in the engineering practice [19].

As an example, let us linearize a nonlinear function of the natural logarithm function given by the following equation [15, 16, 19]:

$$f(x) = \ln(x). \tag{1}$$

It is evident that the arbitrary mathematical function of $f(x)$ centered at $x^* = x$ has no discontinuity at this equilibrium point and is differentiable infinite n times at this point. Thus, the Taylor series of the function given by Equation 1 as follows:

$$
\begin{aligned}
f(x) &= f(x)|_{x=x^*} + f^{'}(x)|_{x=x^*}(x-x^*) + \frac{1}{2}f^{''}(x)|_{x=x^*}(x-x^*)^2 + \cdots + \frac{1}{n!}f^{n}(x)\Big|_{x=x^*}(x-x^*)^n = f(x)|_{x=x^*} \\
&+ \frac{1}{n!}\sum_{n=0}^{\infty} f^{n}(x)|_{x=x^*}(x-x^*)^n = f(x)|_{x=x^*} + f^{'}(x)|_{x=x^*}(x-x^*) + H.O.T.
\end{aligned}
\tag{2}
$$

The first-order approximation of the Taylor series, when sum of the higher order terms meet condition of

$$\sum_{n=2}^{\infty} H.O.T \cong 0 \tag{3}$$

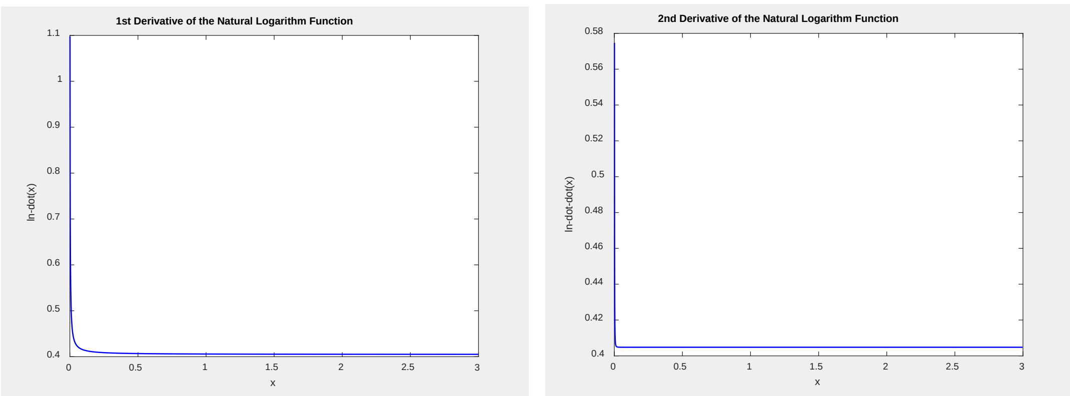
and it leads to the linear approximated function as given below:

$$f_{appr}\,(x) \cong f(x)|_{x=x^*} + f^{'}(x)|_{x=x^*}(x-x^*) \tag{4}$$

Taking multiple derivatives of the natural logarithm function:

$$f(x) = \ln(x);\quad f^{'}(x) = x^{-1};\quad f^{''}(x) = -x^{-2};\quad f^{'''}(x) = 2x^{-3};\quad f^{''''}(x) = -6x^{-4}\ \cdots \tag{5}$$

The first two derivatives of the function $f(x) = \ln(x)$ are depicted in Fig. 1. [17, 18].



(a) (b)

FIG. 1 Derivatives of the natural logarithm function $f(x) = \ln(x)$.

Let us consider for the linearization the following equilibrium point at $x = x^* = 1{,}5$. Thus, the linearized function we seek will have a form as follows:

$$
\begin{aligned}
f_{appr}\,(x) \cong f(x)|_{x=x^*} + f^{'}(x)|_{x=x^*}(x-x^*) &= 0{,}408793 + \frac{1}{1{,}5}(x-1{,}5) \\
&= -0{,}591207 + 0{,}666666 \cdot x
\end{aligned}
\tag{6}
$$

The natural logarithm function and the approximated liner tangential to this function at $x = x^* = 1{,}5$ equilibrium (E) can be seen in Fig. 2. [17, 18].

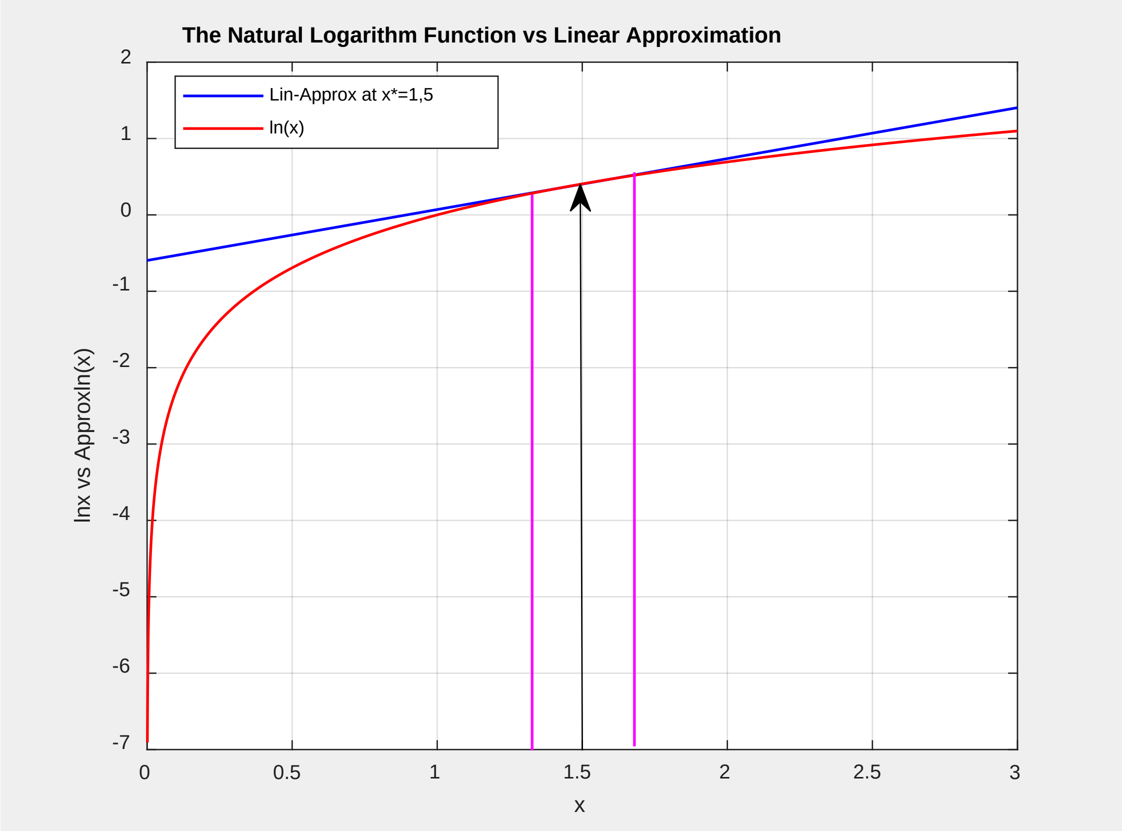


FIG. 2 The natural logarithm function $f(x) = \ln(x)$ and its first order approximation.

The first order linear approximation performs the best at the $x = x^* = 1{,}5$ equilibrium (E) point as a tangential to the natural logarithm function, elsewhere has error due to increase of the sum of the neglected higher order terms (H.O.T.). Errors of the approximation (linearization) calculated for edge points of the range $\Delta x^* = \pm 0{,}15$ are tabulated in Table 1.

Table 1 Approximation error

	x		
	1,35	$x = x^* = 1{,}5$	1,65
Arbitrary function $f(x) = ln(x)$	0,300105	0,408793	0,500775
Linearized function $f_{appr}(x)$	0,308792	0,408793	0,508792
Error of the linearization, $e = f_{appr}(x) - f(x)$	0,008687 ($\approx$ 0,8%)	0	0,008017 ($\approx$ 0,8%)

Leaning on data of Table 1 one can state that although for wide range of x around the equilibrium point (E) the error of the linearization is small, and the linearized (approximated) function is suitable for further applications. As to leave the equilibrium point E for larger values of x as the error of the approximation increases.

Note that error of the linearization is an important issue and set prior the first-order approximation of the arbitrary function $f(x) = ln(x)$.

In other words, knowing the maximums of the error e bounds, the range of x at which the arbitrary function $f(x)$ can be replaced by the linear function $f_{appr}(x)$ can be set.

In engineering sciences (i.e. electrical, mechanical, mechatronics etc.) the decaying exponential function is widely applied.

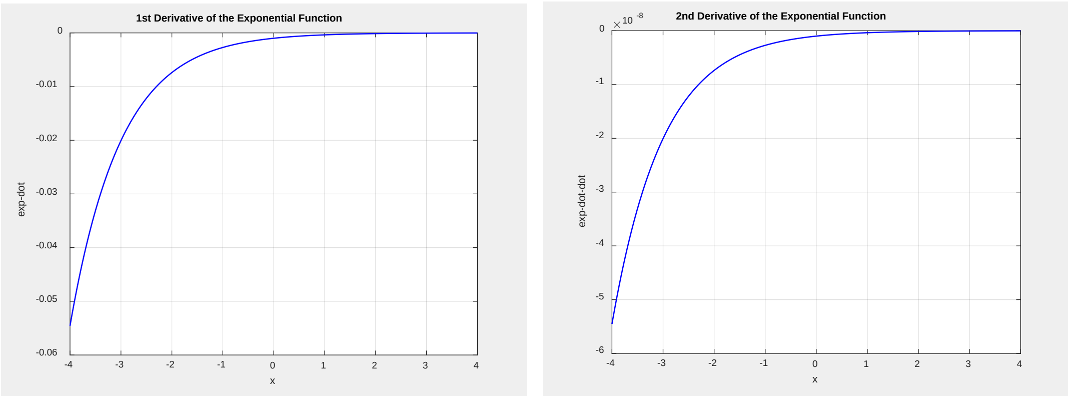
For further studies let us consider the following decaying exponential function expression [16, 19]:

$$f(x) = \mathrm{e}^{-x} \tag{7}$$

Taking multiple derivatives of the exponential function [19]:

$$f(x) = e^{-x};\quad f^{'}(x) = -e^{-x};\quad f^{''}(x) = e^{-x};\quad f^{'''}(x) = -e^{-x},\qquad f^{''''}(x) = e^{-x}\quad \ldots \tag{8}$$

The first two derivatives of the function $f(x) = e^{-x}$ are depicted in Fig. 3. [17, 18].



(a) (b)

FIG. 3 Derivatives of the natural logarithm function $f(x) = \mathrm{e}^{-x}$.

Let us consider for the linearization the following equilibrium point at $x = x^{*} = 0$, when:

$$f(o) = \mathrm{e}^{-0} = 1;\ f^{'}(0) = -\mathrm{e}^{-0} = -1 \tag{9}$$

Thus, the linearized function we seek will have a form as follows:

$$f_{appr}(x) \cong f(x)|_{x=x^{*}} + f^{'}(x)|_{x=x^{*}}(x - x^{*}) = 1 - x \tag{10}$$

The decaying exponential function and the approximated linear tangential to this function at $x = x^{*} = 0$ equilibrium (E) can be seen in Fig. 4. [17, 18].

The first order linear approximation performs the best at the $x = x^{*} = 0$ equilibrium (E) point as a tangential to the natural logarithm function, elsewhere has error due to increase of the sum of the neglected higher order terms (H.O.T.). Errors of the approximation (linearization) calculated for edge points of the range $\Delta x^{*} = \pm 0{,}1$ are tabulated in Table 2.

From Table 2 it is evident that change in x for $\pm 10\%$ will lead to the error of $(\approx 0{,}5\%)$ and $(\approx 0{,}7\%)$, which are acceptable. In case or larger changes in x, the approximation error will increase permanently.

If any accuracy criteria are set, knowing the maximums of the error e bounds, the range of x at which the arbitrary function e^{-x} can be replaced by the linear function $(1 - x)$ can be set.

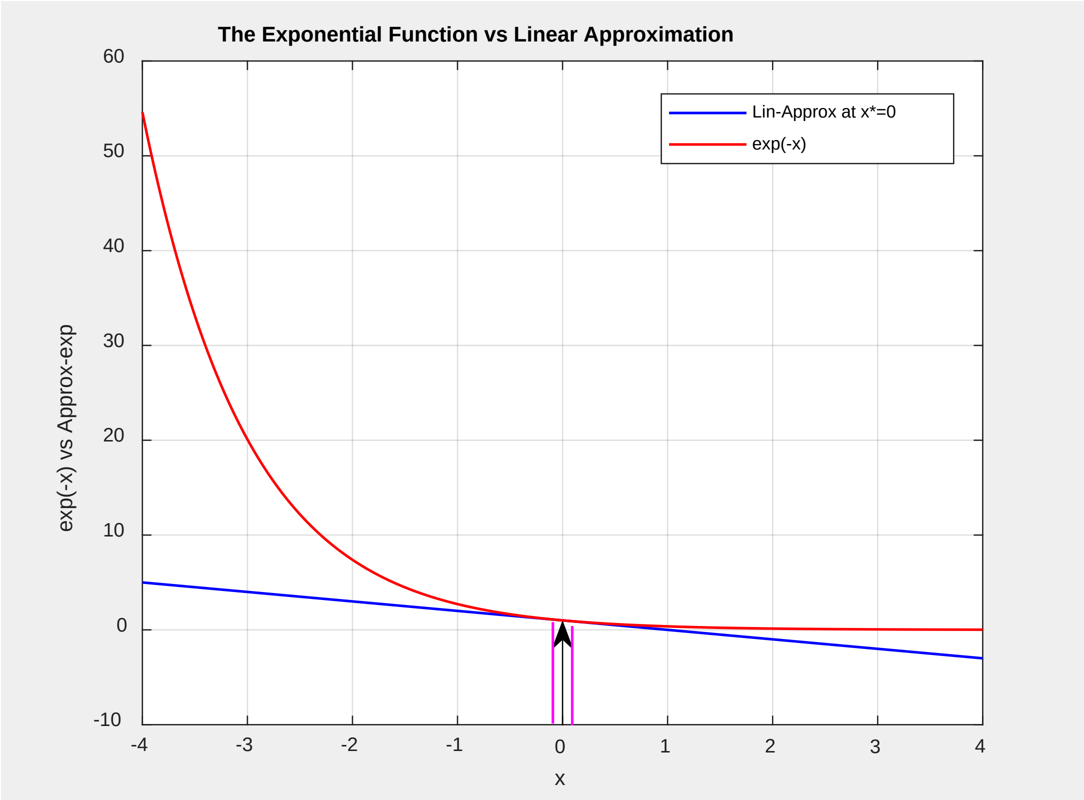


FIG. 4 The decaying exponential function $f(x) = \mathrm{e}^{-x}$ and its first order approximation.

Table 2 Approximation error

	x		
	0,9	$x^* = 0$	1,1
Arbitrary function $f(x) = e^{-x}$	-1,10517	1	0,907556
Linearized function $f_{appr}(x)$	-1,1	1	0,9
Error of the linearization, $e = f(x) - f_{appr}(x)$	0,00517 (≈ 0,5%)	0	0,007556 (≈ 0,7%)

Using mathematical procedure explained above to linearize the exponential function $f(x) = \mathrm{e}^{x}$ one can lean on following derivatives of this function [16, 19]:

$$f(x) = e^{x};\quad f'(x) = e^{x};\quad f''(x) = e^{x};\quad f'''(x) = e^{x},\quad f''''(x) = e^{x}\quad \ldots \tag{11}$$

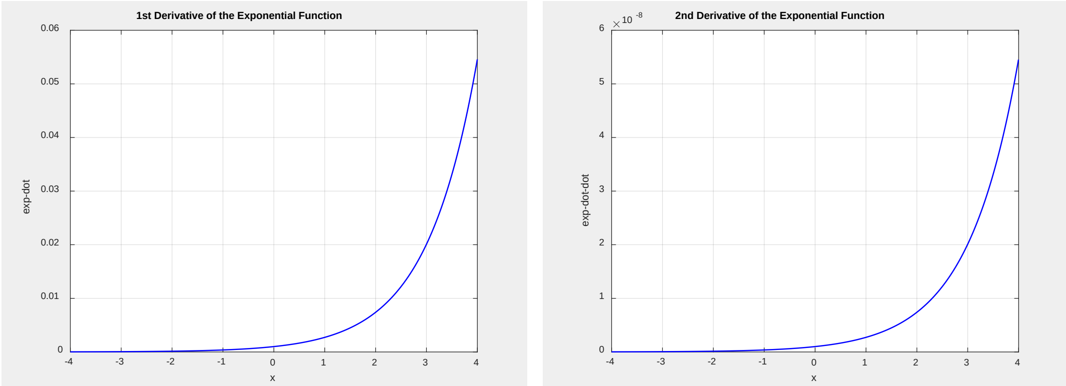
The first two derivatives of the Equation 11 can be seen in Fig. 5. [17, 18].

If to consider for the linearization the following equilibrium point at $x = x^* = 0$:

$$f(o) = e^{0} = 1;\ f'(0) = \mathrm{e}^{0} = 1 \tag{12}$$

Thus, the linearized function we seek will have a form as follows below

$$f_{appr}(x) \cong f(x)|_{x=x^*} + f'(x)|_{x=x^*}(x - x^*) = 1 + x \tag{13}$$



(a) (b)

FIG. 5 Derivatives of the natural logarithm function $f(x) = \mathrm{e}^x$.

The exponential function and the approximated linear tangential to this function at $x = x^* = 0$ equilibrium (E) can be seen in Fig. 6. [17, 18].

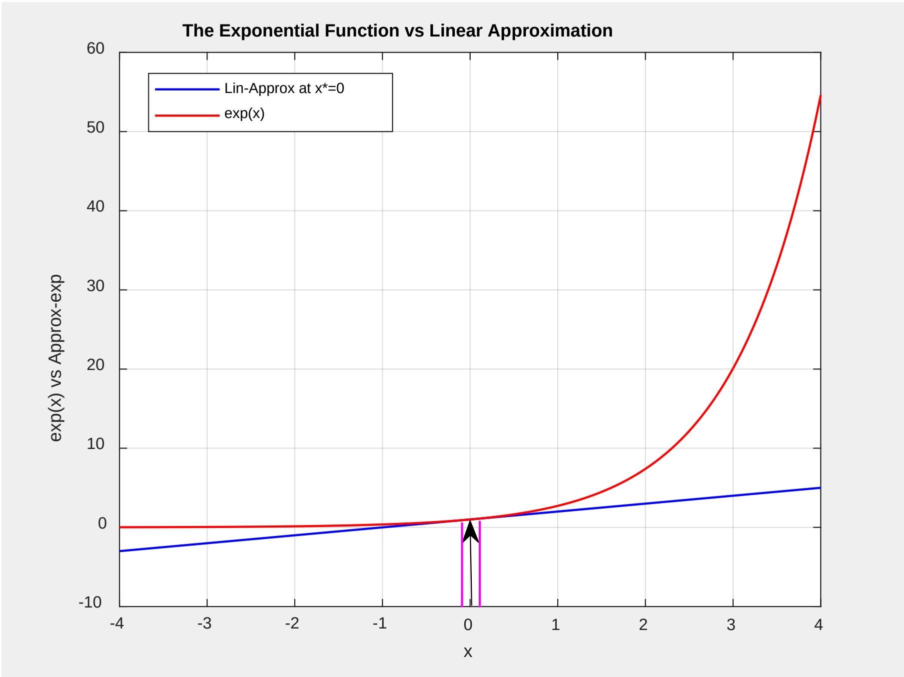


FIG. 6 The exponential function $f(x) = \mathrm{e}^x$ and its first order approximation.

The first order linear approximation performs the best at the $x = x^* = 0$ equilibrium (E) point as a tangential to the natural logarithm function, elsewhere has error due to increase of the sum of the neglected higher order terms (H.O.T.). Errors of the approximation (linearization) calculated for edge points of the range $\Delta x^* = \pm 0{,}1$ are tabulated in Table 3.

Table 3 Approximation error

	x		
	-0,1	$x^* = 0$	0,1
Arbitrary function $f(x) = e^x$	0,903933	1	1,10517
Linearized function $f_{appr}(x)$	0,9	1	1,1
Error of the linearization, $e = f(x) - f_{appr}(x)$	0,003933 ($\approx$ 0,39%)	0	0,00517 ($\approx$ 0,51%)

From Table 3 it is evident that the linearization at the given equilibrium x^*accurate to model the nonlinearity at the newly selected equilibrium with its range defined. Nevertheless, it is easy to agree on that the wider the range selected for x the bigger the error of the approximation will occur.

For further studies and for engineering applications one can apply and use Table 4 for the linearization of the representative nonlinear functions:

Table 4 Representative arbitrary functions and their derivatives

Arbitrary function $f(x)$	First derivative	First-order linear approximation, $f_{appr}\ (x)$
$x^n, \mathbb{R}/\{0\}, n \in \mathbb{Z}$	$n \cdot x^{n-1}$	$2x,$ for $n = 2,\ x^* = 0$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\frac{1}{2} + \frac{1}{2}x,$ for $x^* = 1$
$\frac{1}{x}, \mathbb{R}/\{0\}$	$-\frac{1}{x^2}$	$2 - x,$ for $x^* = 1$
a^x	$a^x \cdot lna$	$1 + ln2 \cdot x,$ for $a = 2,\ x^* = 0$
$log_a x$	$\frac{1}{x \cdot lna}$	$\frac{1}{ln10} \cdot (x - 1),$ for $a = 10,\ x^* = 1$
$sinx$	$cosx$	$x,$ for $x^* = 0$
$cosx$	$-sinx$	$1 - x,$ for $x^* = 0$
$tanx$	$\frac{1}{cos^2 x}$	$1 + \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}(x - \frac{\pi}{4}),$ for $x^* = \pi/4$
$ctanx$	$-\frac{1}{sin^2 x}$	$1 - \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}(x - \frac{\pi}{4}),$ for $x^* = \pi/4$

In inverse problem formulation, any pre-defined error of the accuracy of the linearization can be used for finding the acceptable range of the independent x.

Using the method of the Taylor series expansion dynamic nonlinearities can be handled and linearized modelling the Taylor series with its first-order approximation, when the sum of the H.O.T. is very small compared with the first linear term of the series.

Thus, the H.O.T. representing the error of the linearization (approximation) and it can be omitted without any loss of originality of the arbitrary nonlinear function.

3.2 Time invariant static nonlinearities of UAV closed loop control systems. It is easy to agree that any technical device being involved to solve control problems in the UAVs automatic flight control systems have several static nonlinearities both in its feedforward and feedback paths.

The feedforward path is of controller, amplifier, valves, actuators, famous mostly about their static nonlinearities like dead-zone plus saturation. In case of presence of multiple static nonlinearities, it is supposed that all static nonlinearities can be converted into a single block of the static nonlinearities.

The feedback path sensors often have nonlinearities, in other words, rate limiters introduced by the designers, to limit maneuverability of the UAV, which is extremely important to avoid aggressive UAV maneuvers. If multiple nonlinearities are in the feedback path, it is supposed that they can be blocked into a single unit of the static nonlinearity (Fig. 7).

When to discuss any nonlinear system analysis it is supposed that the linear system dynamics can be decoupled from the static nonlinearities, and all nonlinearities can be unified in one block, as it shown in Fig. 7.

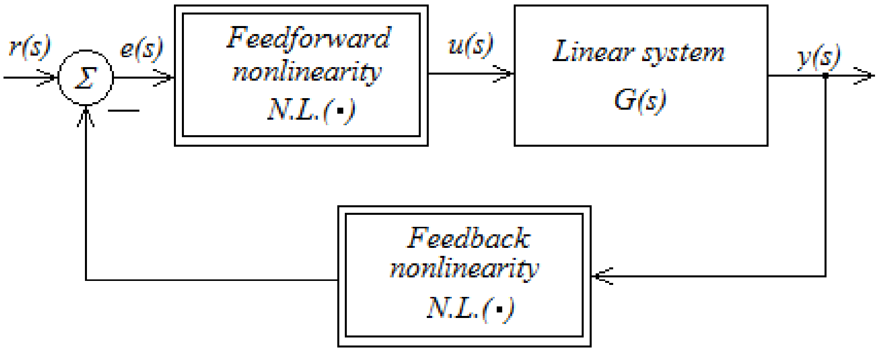


FIG. 7 Decoupling linear dynamics and static nonlinearities.

To model the static nonlinearity, and to find conditions for any limit cycle of the nonlinear closed loop control system describing functions are used widely [5, 6, 7, 8, 9, 10, 11, 12, 13, 15], and the interested reader should refer to.

4. CONCLUSIONS, OUTLOOK AND FUTURE WORK

The linear system is a rare phenomenon in control engineering practice. Although dynamics of the plant to be controlled is a nonlinear awaiting its linearization. This paper mostly focuses on the linearization method called Taylor series expansion. Few functions used the most widely have been linearized and shown that at a properly selected equilibrium points the can be substituted with linear (linearized) functions with tolerable errors.

Errors calculated for the linearization introduced are small for the pre-determined range of the independent. Any increase of the range bounds we linearize over will lead to increase of the linearization error.

In the practice, linearization often starts with definition of the error describing the inaccuracy allowed when to approximate any arbitrary mathematical function with its linear tangential substitution.

Future work will address the nonlinear closed loop control system analysis using describing function methodology to determine closed loop stability and existence of limit cycle generated by the time invariant static nonlinearities. Second field being discussed is a pilot induced oscillation (PIO) of the UAV, when a less-skilled and less-trained amateur operator flies the UAV.

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