SOME REMARKS ON LUPAŞ OPERATORS OF THE SECOND KIND

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***Abstract**: In this paper we give a probabilistic interpretation of Lupaş operators of the second kind and a quantitative estimation of theirs.*

***Keywords**: Gamma type operators, Rathore operators, Szasz-Mirakjan operators, Lupaş operator of the second kind, probabilistic interpretation.*

1. INTRODUCTION

The operators of gamma type were studied by many authors (Lupaş, 1967, 1995a, 1995b; Muller, 1967; Rathore, 1973; Miheşan, 2003). In this paper, we consider Rathore operators (1973) which are gamma type operators and are defined by

$$\begin{cases} R_t(f;x) = E\left[f\left(\dfrac{U_{t\,x}}{t}\right)\right] = \dfrac{1}{\Gamma(t\,x)}\displaystyle\int_0^\infty e^{-u}u^{t\,x-1} f\left(\dfrac{u}{t}\right)du, & t>0,\ x>0 \\ R_t(f;0) = f(0) \end{cases} \quad (1)$$

These operators were given for $t = n,\ n \in N$ but they are well- defined for $t > 0$, if f is a real measurable function on $[0,\infty)$ for which, the mean value of the random variable $\left|f\left(\dfrac{U_{t\,x}}{t}\right)\right|$ exists, i.e. $E\left[\left|f\left(\dfrac{U_{t\,x}}{t}\right)\right|\right] < \infty$, where $\{U_{t\,x} : t \geq 0,\ x \geq 0\}$ is a gamma stochastic process with probability density :

$$\rho_{t\,x}(u) = \begin{cases} \dfrac{u^{t\,x-1}e^{-u}}{\Gamma(tx)}, & t>0,\ x>0,\ u>0 \\ 0, & t=0,\ x=0,\ u=0 \end{cases} \quad (2)$$

If $U_{t\,x}$ has the probability density $\rho_{t\,x}(u)$ then $V_{t\,x} = \dfrac{U_{t\,x}}{t}$ has the probability density (Feller, 1966:53)

$$g_{t\,x}(u) = t\rho_{t\,x}(tu) = \frac{t}{\Gamma(t\,x)} e^{-t\,u}(t\,u)^{t\,x-1}, \quad t>0,\ x>0 \quad (3)$$

Indeed, by a change of variable, we have for Rathore operators the representation

$$\begin{cases} R_t(f;x) = E[f(V_{t\,x})] = \dfrac{1}{\Gamma(t\,x)}\displaystyle\int_0^\infty e^{-t\,u}(t\,u)^{t\,x-1} f(u)du \\ R_t(f;0) = f(0) \end{cases} \quad (4)$$

It is known that, these operators are linear positive operators which preserve the linear functions:

$$\begin{cases} R_t(e_0;x) = e_0(x) = 1 \\ R_t(e_1;x) = e_1(x) = x,\ x \geq 0,\ t>0,\ e_i(x) = x^i,\ i = 0, 1, 2 \\ R_t(e_2;x) = e_2(x) = x^2 + \dfrac{x}{t} \end{cases} \quad (5)$$

and have the same properties as Szasz-Mirakjan operators

$$S_t(f;x) = E\left[f\left(\frac{N_{t\,x}}{t}\right)\right] = e^{-t\,x}\sum_{k=0}^{\infty}\frac{(t\,x)^k}{k!} f\left(\frac{k}{t}\right), \quad x \geq 0,\ t>0 \quad (6)$$

On these operators (6), $\{N_{t\,x} : t \geq 0,\ x \geq 0\}$ is a standard Poisson stochastic process, independent of the gamma process (2) but defined on the same probability space as former and for which $E\left[f\left(\dfrac{N_{t\,x}}{t}\right)\right]$ represent the mean value of random variable $\left[f\left(\frac{N_{t\,x}}{t}\right)\right]$.

Now, we consider the Lupaş operators of second kind [4] which are linear and positive operators and preserve the linear functions defined as

$$L_t(f;x)=\begin{cases}2^{-t\,x}\sum_{k=0}^{\infty}\frac{(t\,x)_k}{2^k\,k!}f\left(\frac{k}{t}\right), & x>0,\ t>0\\ f(0), & x=0\end{cases} \qquad (7)$$

$f\in C_B[0,\infty)$,

where

$(t\,x)_k=(t\,x)(t\,x+1)(t\,x+2)\ldots(t\,x+k-1)$, $k\in N$, is the k – order rising factorial power of *tx*,. Some approximation properties of these operators were given in Lupaş, 1995, Agratini, 2000.

Using a gamma first kind transformation (Miheşan, 2003:49-54)

$$\Gamma_p^a(f;x)=\frac{1}{\Gamma(p)}\int_0^{\infty}e^{-t}t^{p-1}f\left(x\left(\frac{t}{p}\right)^a\right)dt,\quad a\in R,\ p>0,\ x>0$$

,is easy to see that, for $a=1$ together with Miheşan (2003:49-54, th.2.4.), Lupas operators can be represented as a combination between Rathore operators (1) and Szasz-Mirakjan operators (6)

$$L_t(f;x)=R_t(S_t)(f;x) \qquad (8)$$

With this representation we have

$$L_t(e_i;x)=e_i(x)=x^i,\ i=\overline{0,1}$$

$$L_t(e_2;x)=x^2+\frac{2\,x}{t},\ x\ge 0,\ t>0$$

Remark.

$$\begin{cases}L_{r\,t}\left(f(t\,u);\frac{x}{t}\right)=L_r(f;x)\\ L_r\left(f\left(\frac{u}{t}\right);t\,x\right)=L_{r\,t}(f;x)\end{cases}$$

$r>0,\ t>0,\ u>0,\ x\ge 0$.

2. PROBABILISTIC INTERPRETATION OF LUPAŞ OPERATORS OF THE SECOND KIND

In view that, for the kernel of Lupaş operators we obtain randomizing the parameters of Poisson distribution according to gamma distribution (Feller, 1966:52-53) for $t>0,\ x>0$ with (3), that:

$$p_{t,k}(x)=P\left(N_{V_{t\,x}}=k\right)=$$

$$=\frac{t}{\Gamma(t\,x)}\int_0^{\infty}e^{-t\,u}\frac{(t\,u)^k}{k!}e^{-t\,u}(t\,u)^{t\,x-1}du=$$

$$=\frac{t}{k!\Gamma(t\,x)}\int_0^{\infty}e^{-2t\,u}(t\,u)^k(t\,u)^{t\,x-1}du=$$

$$=\frac{t}{k!\Gamma(t\,x)}\int_0^{\infty}e^{-\theta}\frac{\theta^{t\,x+k-1}}{2^{t\,x+k}}d\theta=\frac{\Gamma(t\,x+k)}{k!2^{t\,x+k}\Gamma(t\,x)}=$$

$$=\binom{t\,x+k+1}{k}\frac{1}{2^{t\,x+k}}$$

So the operator $L_t(f;x)=E\left[f\left(\frac{N_{V_{t\,x}}}{t}\right)\right]$ is the mean value of the random variable $f\left(\frac{N_{V_{t\,x}}}{t}\right)$, for any measurable function f for which $E\left[\left|f\left(\frac{N_{V_{t\,x}}}{t}\right)\right|\right]<\infty$.

An interesting result, which was obtained by De la Cal & Carcamo (2007:1106-1115) for the operators of Bernstein – type which preserves the affine functions, namely "centered Bernstein-type operators", can be used for these Lupaş operators of the second kind :

Theorem 2.1. (De la Cal & Carcamo, 2007:1106-1115). If $L_1=L_2\circ L_3$, where L_1, L_2, L_3 are centered Bernstein-type operators ($Lf(x)=E[f(Y_x)],\ x\in I\subset R,$ $Le_1(x)=E[Y_x]=x$) on the same interval I and if $\mathcal{L}_{cx}$ is the set of all convex functions in the domain of the three operators, then $L_1f\ge L_2f,\ f\in\mathcal{L}_{cx}$. If in addition L_3 preserves convexity, then $L_1f\ge L_2f\vee L_3f,\ f\in\mathcal{L}_{cx}$ where $f\vee g$ denotes the maximum of f and g.

Indeed, using the representation (7) for Lupaş operators of the second kind, we have for $t > 0,\ L_t f \geq R_t f,\ f \in \mathcal{L}_{cx}[0, \infty)$ and $L_t f \geq R_t f \vee S_t f,\ f \in \mathcal{L}_{cx}[0, \infty)$, where S_t are Szasz-Mirakjan operators (6), R_t are Rathore operators (1) and L_t are Lupaş operators of the second kind (7).

An estimation of the difference $|L_t(f;x) - R_t(f;x)|$ is given next.

3. A QUANTITATIVE ESTIMATION

Theorem 3.1. If $f \in C_B[0,\infty)$, then for every $x \in [0, \infty)$ and $t > 0$ we have

$$|L_t(f;x) - R_t(f;x)| \leq \left(1 + \delta^{-2}\frac{x}{t}\right)\varpi_f(\delta).$$

Proof. Using an estimation of Shisha & Mond (1968) with the modulus of continuity of f relative to Szasz-Mirakjan operators (6) $|S_t(f;x) - f(x)| \leq \left(1 + \delta^{-2}\frac{x}{t}\right)\varpi_f(\delta)$, we have with (8) that

$$|L_t(f;x) - R_t(f;x)| = |R_t(S_t(f;x)) - R_t(f;x)| \leq$$
$$\leq \frac{1}{\Gamma(tx)}\int_0^\infty e^{-\theta}\theta^{tx-1}\left|S_t f\left(\frac{\theta}{t}\right) - f\left(\frac{\theta}{t}\right)\right| d\theta \leq$$
$$\leq \frac{1}{\Gamma(tx)}\int_0^\infty e^{-\theta}\theta^{tx-1}\left(1 + \delta^{-2}\frac{\theta}{t^2}\right)\varpi_f(\delta)d\theta \leq$$
$$\leq \left(1 + \delta^{-2}\frac{\theta}{t^2}\right)\varpi_f(\delta)$$

So, we obtain the same estimation for the difference

$$|L_t(f;x) - R_t(f;x)| = |R_t(S_t(f;x)) - R_t(f;x)|$$

as for the difference $|S_t(f;x) - f(x)|$.

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