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THE DEVELOPING MILITARY ROBOTICS

Prof. Dr. SZABOLCSI, Róbert

Bolyai János Faculty of Military Technology, Zrínyi Miklós National Defense University

Abstract: *Robots are widely used tools in remote sensing, and other fields. Based on theirs abilities robots are able to carry on tasks unbelievable fir human beings. Main problem proposed for solution by the author is to solve problems of automation of the flight of the UAV systems. In many air robot applications there is a need from the users to automatize flight of the aircraft increasing flight safety, and, quality of the control of the UAV systems. From among those available air robots this article deals with quadrotors. The closed loop control systems of the UAV being investigated is control of the vertical position of the UAV. The vertical motion will be controlled with LQR controller so as to provide flying and handling quality of the UAV.*
Keywords: *military robots, recce surface robots, air robot systems, CAD.*

1. INTRODUCTION

There is no doubt that number of robots designed by state-of-the technologies used for both military and non-military purposes is significantly increasing. Fields of possible applications of the robots and robot systems are as follows: investigation of the climate changes; air reconnaissance of flooding; elimination of consequences of the disasters; industrial catastrophes and theirs control; urban security applications; urban control of traffic of public transport; surface recce missions using ground robots; underwater applications; recce of dangerous areas; recce about safety and security items; solution of problems of defense of the critical infrastructure. Main purpose of the author is to design an optimal controller for the UAV being controlled along its vertical axes, i.e. vertical motion is controlled.

2. LITERATURE REVIEW

Mathematical models of the dynamical systems are outlined in [2,3,4,10]. They deal with analysis and design problems. Theoretical backgrounds of the automatic flight control systems are in [8], providing large scale of aircraft models, and giving examples for optimal control of aircraft. UAV systems are investigated in [1,5,6,12,13,]: there are many analysis and design examples applied to present latest results of robotics, mechatronics, and, sensorics. In [14] dynamic performances are summarized, and used in this paper. In [9] dynamic performances and stability analysis is shown for micro UAV: a complex task is made for deriving dynamical model of micro rotating UAV. Pokorádi in [11] deals with deterministic signals applied in control system analysis. Computer-aided design and analysis is supported by computer package MATLAB [3, 7].

3. DYNAMIC MODEL OF THE QUADRTOTORS

The quadrotor dynamics may be analyzed using Figure 1. [1,5,6,9,12,13]. Maneuvering along vertical axis is a common task, e.g. change height of the flight from the initial hovering position. The coordinate system **I** represent the system of inertia, the body-axis system centre is fixed in point **B**.

The rate of changes of the Euler-angles in the body-axis system may be derived as [12]:

$$\begin{aligned}\begin{bmatrix}\dot{\varphi} & \dot{\theta} & \dot{\psi}\end{bmatrix}^T &= \mathbf{M}^{-1}\begin{bmatrix}\omega_{x_i} & \omega_{y_i} & \omega_{z_i}\end{bmatrix}^T = \\ &= \mathbf{M}^{-1}\mathbf{A}\begin{bmatrix}\omega_{x_b} & \omega_{y_b} & \omega_{z_b}\end{bmatrix}^T\end{aligned} \quad (3.1)$$

where: ϕ is bank angle; θ pitch angle; ψ yaw angle; ω_{x_i} angular rates of changes in the inertia system **I**, ω_{x_b} are angular rates of changes in the body-axis coordinate system. It is well-known that rotational matrices between two coordinate systems given above are as follows [12]:

$$\mathbf{M} = \begin{bmatrix} \frac{c\psi}{c\theta} & \frac{s\psi}{c\theta} & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi - s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix},$$

where: c – cos, s – sin.

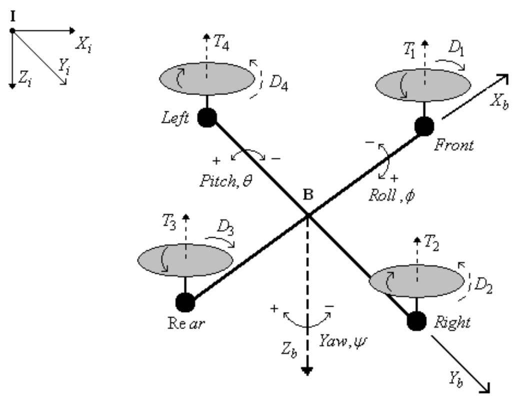

Figure 1. Quadrotor dynamics and kinematics.

For further investigations we will consider dynamics of the quadrotor in the body axis system, the following matrix equation must be defined:

$$[\dot{x}_b \quad \dot{y}_b \quad \dot{z}_b]^T = \mathbf{A}^{-1}[\dot{x}_i \quad \dot{y}_i \quad \dot{z}_i]^T, \quad (3.2)$$

where x_b, y_b, z_b are coordinates in the body-axis coordinate system, and x_i, y_i, z_i are the coordinates measured in the inertia system.

3.1 Equations of motion of the translational motion of the quadrotor. It is assumed that quadrotor has rigid and symmetric airframe and we will consider it with its mass concentrated in point B (see Figure 1.). Quadrotor owns rigid rotor blades, and finally, there is considered only motion along vertical axis of body-axis coordinate system Z_b.

The lift force generated by the i^{th} rotor blades is proportional to its squared rotational speed, i.e. we have [5, 12]:

$$T_i = C_1\left(\frac{1-2\pi LCS}{P\alpha_i} + 2\pi\frac{\dot{z}_b - w_{z_b}}{P\alpha_i}\right), \quad (3.3)$$

where: $C_1 = k_t \rho A_p \alpha_i^2 R_p^2$; k_t is aerodynamic coefficient; ρ is air density; A_p is resulting area of the rotor blades; α_i is angular speed of the i^{th} rotor blades; R_p is radius of the rotor blades; L is a distance measured between centre point of the blades and point **B**; P is a setting angle of the rotor blades; w_{z_b} is a component vector of the atmospheric turbulence projected to vertical axis. It is obvious that: C=1, if i=1, or i=4; C= –1, if i=2, or i=3; $S = \omega_{y_b}$, if i=1, or i=3; $S = \omega_{x_b}$, if i=2, or i=4 [5,6,12].

The resulting force acting along longitudinal axis of the quadrotor may be derived as:

$$\mathbf{F}_{wI} = \mathbf{A}\left[k_s(w_{x_b} - \dot{x}_b) \quad k_s(w_{y_b} - \dot{y}_b) \quad k_u(w_{z_b} - \dot{z}_b)\right]^T, \quad (3.4)$$

where: k_s, k_u friction coefficients; w_{x_b} and w_{y_b} are components of the speed of the turbulent air along axis x–, and y–, respectively.

The quadrotor translational motion state equation may be derived as follows below:

$$\begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{z}_i \end{bmatrix} = -\begin{bmatrix} \omega_{x_b} \\ \omega_{y_b} \\ \omega_{z_b} \end{bmatrix} \times \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{bmatrix} + \mathbf{g}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{F_{wI}}{m} - \frac{T_1 + T_2 + T_3 + T_4}{m}\mathbf{A}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad (3.5)$$

where $\boldsymbol{g}$ is the gravitational acceleration and m is the mass of the UAV being investigated.

3.2 Equations of motion of the rotational motion of the quadrotor. It is well-known that drag moment of the rotor blades due to its rotational motion is proportional to the speed of its revolution, i.e. one can derive that:

$$D_i = C_2\left(\frac{1-2\pi LCS}{P\alpha_i} + 2\pi\frac{\dot{z}_b - w_{z_b}}{P\alpha_i}\right), \quad (3.6)$$

where: $C_2 = k_d \rho A_p \alpha_i^2 R_p^3$; k_d is the moment coefficient. The resulting reaction moment of the rotor blades may be derived using following equation:

$$I_{ct} = J_p\left(-\dot{\alpha}_1 + \dot{\alpha}_2 - \dot{\alpha}_3 + \dot{\alpha}_4\right), \tag{3.7}$$

where J_p is the moment of inertia of a single rotor blade.

The friction drag moment may be derived as follows below:

$$\mathbf{M}_f = k_r\left[\dot{\phi} \quad \dot{\theta} \quad \dot{\psi}\right]^T, \tag{3.8}$$

where: k_r is a friction coefficient. The resulting disturbances (e.g. gust) related to DC-motor rotors may be found using following formula [16]:

$$\tau_d = \left[\tau_{x_b} \quad \tau_{y_b} \quad \tau_{z_b}\right]^T. \tag{3.9}$$

The gyroscopic moment may be derived as:

$$\mathbf{M}_g = J_p\left[\dot{\theta}\alpha \quad \dot{\phi}\alpha \quad 0\right]^T, \tag{3.10}$$

where: $\alpha = -\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4$.

Finally, using equations defined above, the state equation of the quadrotors spatial rotational motion may be derived as [5,6,12]:

$$\begin{bmatrix} \dot{\omega}_{x_b} \\ \dot{\omega}_{y_b} \\ \dot{\omega}_{z_b} \end{bmatrix} = -J^{-1}\omega \times J \begin{bmatrix} \omega_{x_b} \\ \omega_{y_b} \\ \omega_{z_b} \end{bmatrix} - J^{-1}\left(\mathbf{M}_f + \tau_d + \mathbf{M}_g\right) + J^{-1}\begin{bmatrix} L(T_4 - T_2) \\ L(T_1 - T_3) \\ D_1 - D_2 + D_3 - D_4 + I_{ct} \end{bmatrix}, \tag{3.11}$$

where: $\omega = \begin{bmatrix} 0 & -\omega_{z_b} & \omega_{y_b} \\ \omega_{z_b} & 0 & -\omega_{x_b} \\ -\omega_{y_b} & \omega_{x_b} & 0 \end{bmatrix}$, $J = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix}$

is a main matrix of inertia; J_{xx}, J_{yy}, J_{zz} are moments of inertia related to axes X_b, Y_b, and Z_b, respectively.

3.3 Dynamics of the quadrotor DC-motor. DC motors (mainly brushless ones) are widely applied in propulsion systems of quadrotors. So, theirs equation – for small inductances – may be derived as [5,6,12]:

$$J_p\dot{\alpha}_i = G\tau_{m_i} - D_i, \tag{3.12}$$

where $\tau_{m_i} = \dfrac{k_i(V_i - \dfrac{k_v\alpha_i}{G})}{R}$ is the motor moment; k_i is a motor constant; k_v is a motor constant for rotation speed; V_i is a motor control voltage; R is a motor resistance; G is a constant transmission gain of the system "motor-rotor blade".

Let us find dynamics of the quadrotor in motion along vertical axis, for the initial conditions defined as:

$$\theta = 0^o; \phi = 0^o; \psi = 0^o; v_{x_{b_o}} = 0m/s; \; y_{b_o} = 0m/s; v_{z_{b_o}} = 0m/s. \tag{3.13}$$

Using equations (3.1)–(3.5), and considering initial conditions of (3.13) translational motion of the quadrotor along vertical axis may be derived as:

$$\ddot{z}_b = \frac{F_{mI}}{m} - \frac{T_1 + T_2 + T_3 + T_4}{m} + g, \tag{3.14}$$

or, in other manner

$$\ddot{z}_b + \frac{\dot{z}_b}{m} = g - \frac{T_1 + T_2 + T_3 + T_4}{m} = g - \frac{4T}{m}, \tag{3.15}$$

Lift generated by rotor blades may be derived as:

$$T = C_1\left(\frac{1}{P\alpha_i} + 2\pi\frac{\dot{z}_b}{P\alpha_i}\right), \tag{3.16}$$

where: $C_1 = k_t\rho A_p\,\alpha_i^2 R_p^2 = 4{,}15872 \cdot 10^{-6}\,\alpha_i^2$.

Let us substitute equation (3.16) into equation (3.15), it yields to:

$$\ddot{z}_b + \frac{\dot{z}_b}{m} = g - \frac{4T}{m} = g - \frac{4}{m}C_1\left(\frac{1}{P\alpha_i} + 2\pi\frac{\dot{z}_b}{P\alpha_i}\right), \tag{3.17}$$

and rearranging equation (3.17), one may write:

$$\ddot{z}_b + \frac{\dot{z}_b}{m} + \frac{4}{m}C_1 2\pi\frac{\dot{z}_b}{P\alpha_i} = g - \frac{4}{m}C_1\frac{1}{P\alpha_i}, \tag{3.18}$$

and, finally, doing some mathematical arrangements, we get following formula:

$$\ddot{z}_b + \dot{z}_b\left(\frac{1}{m} + \frac{4}{m}C_1 2\pi\frac{1}{P\alpha_i}\right) = g - \frac{4}{m}C_1\frac{1}{P\alpha_i} . \quad (3.19)$$

Using hypothetical quadrotor data given in [16] following equation of motion may be derived:

$$z_b + z_b\left(0{,}222568 + 153{,}0451369 \cdot 10^{-6}\alpha_i\right) = \\ = 9{,}81 - 24{,}35789 \cdot 10^{-6}\alpha_i \quad (3.20)$$

Let speed of rotation of DC-motors be the following: $\alpha_{i_o} = 1000 rev/\min$. Thus, equation (3.20) may be rewritten in the following manner:

$$\dot{v}_b + v_b 153{,}2677049 = \\ = 9{,}81 - 24{,}35789\Delta\alpha_i . \quad (3.21)$$

Using equation (3.21) transfer function of the UAV may be derived as [12]:

$$Y(s) = \frac{v_b(s)}{\Delta\alpha_i(s)} = -\frac{24{,}35789}{153{,}2677049 + s} . \quad (3.22)$$

For further investigations UAV model was analyzed in time domain, and in frequency domain. Results of the computer simulation may be seen in Figure 2., and Figure 3 [3, 7].

From Figure 2 it is evident that reaction of the quadrotor is fast. The impulse response function is derives, that the open loop UAV is the stable one.

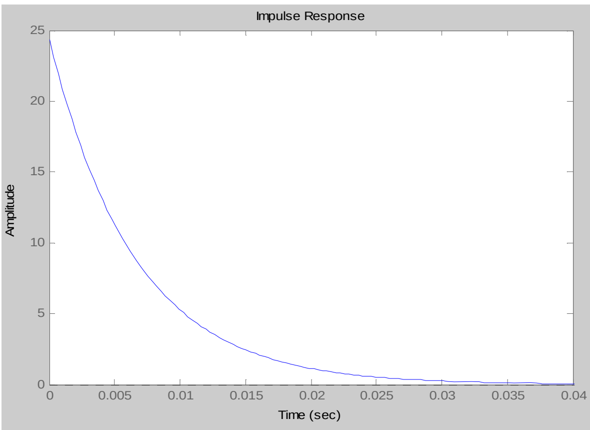

Figure 2a. Results of the Time Domain Transient Response.

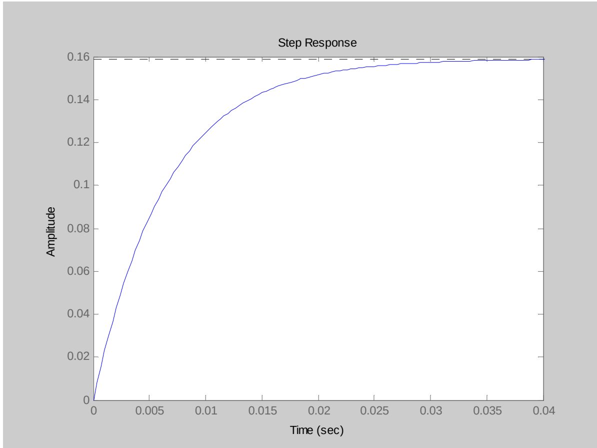

Figure 2b. Results of the Time Domain Transient Response.

Step response function shows that at the end of very fast response, the UAV will start to maneuver along vertical axis, and starts to ascend with constant speed of, say, approximately, 0,16 m/s. Step response function also predicts stable UAV behavior.

Figure 3. describes the frequency domain behavior of the UAV.

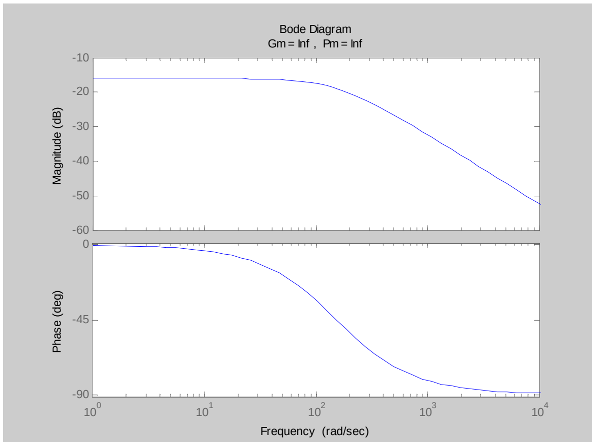

Figure 3. Bode Diagram of the UAV.

From Figure 3 it is evident that the quadrotor has low-pass behavior, in high frequency ranges it is cutting off signals.

4. LQ-BASED CONTROLLER SYNTHESIS FOR VERTICAL SPEED OF THE QUADROTOR

The linear, multi input, multi output (MIMO) system dynamics may be defined using state, and output equation given below [2, 4, 8, 12]:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} ; \mathbf{y} = \mathbf{Cx} + \mathbf{Du} , \quad (4.1)$$

where x is state vector, **u** is the input vector, **y** is the output vector, **A** is the state matrix, **B** is the input matrix, **C** is the output matrix, and finally, **D** is the direct feedforward matrix.

For MIMO control systems integral criteria to be minimized may be derived as [2, 4, 8]:

$$J = \frac{1}{2}\int_0^{\infty}\left(\mathbf{x}^{\mathbf{T}}\mathbf{Q}\,\mathbf{x} + \mathbf{u}^{\mathbf{T}}\mathbf{R}\,\mathbf{u}\right)dt \to \mathrm{Min}, \qquad (4.2)$$

where **Q** and **R** are positive semi-definite, and positive definite, diagonal weighting matrices, respectively.

The term $\mathbf{x}^{\mathbf{T}}\mathbf{Q}\,\mathbf{x}$ in equation (4.2) defines dynamic performances, while term $\mathbf{u}^{\mathbf{T}}\mathbf{R}\,\mathbf{u}$ describes costs. These terms are quadratic ones, because of following formulas:

$$\mathbf{x}^{\mathbf{T}}\mathbf{Q}\,\mathbf{x} = [x_1 \; . \; . \; . \; x_n]\begin{bmatrix} q_1 & 0 & . & 0 & 0 \\ 0 & q_2 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & q_{n-1} & 0 \\ 0 & 0 & . & 0 & q_n \end{bmatrix}\begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = , \qquad (4.3)$$

$$= [x_1 \; . \; . \; . \; x_n]\begin{bmatrix} q_1 x_1 \\ . \\ . \\ . \\ q_n x_n \end{bmatrix} = \sum_{i=1}^{n} q_i\, x_i^2(t)$$

and

$$\mathbf{u}^{\mathbf{T}}\mathbf{R}\,\mathbf{u} = [u_1 \; . \; . \; . \; u_n]\begin{bmatrix} r_1 & 0 & . & 0 & 0 \\ 0 & r_2 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & r_{n-1} & 0 \\ 0 & 0 & . & 0 & r_n \end{bmatrix}\begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_n \end{bmatrix} = . \qquad (4.4)$$

$$= [u_1 \; . \; . \; . \; u_n]\begin{bmatrix} r_1 u_1 \\ . \\ . \\ . \\ r_n u_n \end{bmatrix} = \sum_{j=1}^{n} r_j\, u_j^2(t)$$

Using equations (4.3) and (4.4) it is easy to be seen that integral performance criteria minimizes integrals from squared functions of those $x_i^2(t)$ and $u_j^2(t)$.

4.1 The algebraic Ricatti equation (ARE). It is supposed that state equation of the dynamic system is given as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B} \; . \qquad (4.5)$$

The optimal control law is given as

$$\mathbf{u}^{\mathbf{o}}(t) = -\mathbf{K}\,\mathbf{x}(t), \qquad (4.6)$$

which minimizes integral criterion (4.2). The optimal control is solved for any initial condition of $\mathbf{x}(0)$, if static feedback gain matrix **K** is derived. Block diagram of the optimal control system is given in Figure 4. Let reference the signal be zero value one, i.e. $x_r(t) = 0$.

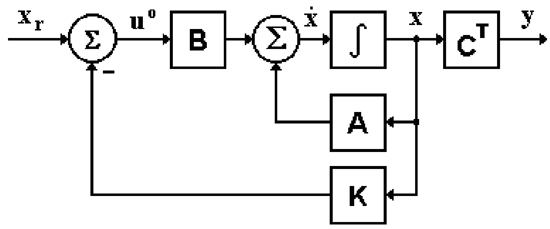

Figure 4. Block Diagram of the Optimal Control System.

Substituting equation (4.6) into equation (4.5) results in the following formula

$$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BKx} = (\mathbf{A} - \mathbf{BK})\mathbf{x} \, . \qquad (4.7)$$

Supposing that matrix $(\mathbf{A} - \mathbf{BK})$ has eigenvalues with negative real parts. Substituting equation (4.7) into equation (4.2) yields to:

$$\begin{aligned} J &= \frac{1}{2}\int_{\mathbf{0}}^{\infty}(\mathbf{x}^{\mathrm{T}}\mathbf{Qx} + \mathbf{x}^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\mathbf{RKx})\,\mathrm{dt} = \\ &= \frac{1}{2}\int_{\mathbf{0}}^{\infty}\mathbf{x}^{\mathrm{T}}(\mathbf{Q} + \mathbf{K}^{\mathrm{T}}\mathbf{RK})\mathbf{x}\,\mathrm{dt} \to \mathrm{Min} \end{aligned} \; . \qquad (4.8)$$

For minimization of the integral performance criterion (4.2) we will use second method of Ljapounov. It is supposed that for any state vector exists a positive definite Hermite-matrix, **P**, so that take place $\mathbf{P} = \mathbf{P}^{\mathbf{T}}$. For this particular case takes place following condition:

$$\mathbf{x}^{\mathrm{T}}(\mathbf{Q} + \mathbf{K}^{\mathrm{T}}\mathbf{RK})\mathbf{x} = -\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{x}^{\mathrm{T}}\mathbf{Px}) \, . \qquad (4.9)$$

Taking derivative of matrix $\mathbf{x}^{\mathbf{T}}\mathbf{P}\,\mathbf{x}$, and considering equation (4.9) results in:

$$\begin{aligned} \mathbf{x}^{\mathrm{T}}(\mathbf{Q} + \mathbf{K}^{\mathrm{T}}\mathbf{RK})\mathbf{x} &= -\,\dot{\mathbf{x}}^{\mathrm{T}}\mathbf{Px} - \mathbf{x}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{x}} = \\ &= -\,\mathbf{x}^{\mathrm{T}}\left[(\mathbf{A} - \mathbf{BK})^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A} - \mathbf{BK})\right]\mathbf{x} \end{aligned} \qquad (4.10)$$

Using second method of, if matrix $(\mathbf{A}-\mathbf{BK})$ has eigenvalues with negative real parts, than for positive definite matrix $\mathbf{Q}+\mathbf{K}^{\mathrm{T}}\mathbf{RK}$ exists positive definite matrix **P**, such that takes place following equation:

$$(\mathbf{A}-\mathbf{BK})^{\mathbf{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK})=-(\mathbf{Q}+\mathbf{K}^{\mathbf{T}}\mathbf{RK}) \quad (4.11)$$

Equation (4.11) is known as Ljapounov equation. The integral performance index may be derived as:

$$J=\frac{1}{2}\int_{\mathbf{0}}^{\infty}\mathbf{x}^{\mathrm{T}}(\mathbf{Q}+\mathbf{K}^{\mathrm{T}}\mathbf{RK})\mathbf{x}\,dt=-\left[\mathbf{x}^{\mathrm{T}}\mathbf{Px}\right]_{\mathbf{0}}^{\infty}= \quad .(4.12)$$
$$=-\mathbf{x}^{\mathrm{T}}(\infty)\mathbf{Px}(\infty)+\mathbf{x}^{\mathrm{T}}(0)\mathbf{Px}(0)$$

Supposing that all eigenvalues of matrix $\mathbf{A}-\mathbf{BK}$ have negative values, thus takes place that $\mathbf{x}(\infty)\rightarrow 0$. In this case equation (4.12) may be rewritten in the following manner:

$$\mathrm{J}=\mathbf{x}^{\mathbf{T}}(0)\mathbf{Px}(0). \quad (4.13)$$

From equation (4.13) it is easily may be seen that integral criteria (4.12) is a function of the initial conditions of **x**(0). It is known that weighting matrix **R** is positive definite, Hermite-matrix, i.e. takes places following equation:

$$\mathbf{R}=\mathbf{T}^{\mathrm{T}}\mathbf{T}, \quad (4.14)$$

where **T** is a non-singular (regular) matrix.

Considering equation (4.14) equation (4.11) may be rewritten as follows below:

$$\begin{aligned}&(\mathbf{A}^{\mathrm{T}}-\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}})\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK})+\\&+\mathbf{Q}+\mathbf{K}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{TK}=0\end{aligned} \quad . \quad (4.15)$$

Rearranging equation (4.15) yields to the following formula:

$$\begin{aligned}&\left(-\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}-\mathbf{PBK}+\mathbf{K}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{TK}\right)+\\&+\mathbf{Q}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{PA}=0\end{aligned} \quad . \quad (4.16)$$

It is known that $\mathbf{P}=\mathbf{P}^{\mathbf{T}}$, and $\mathbf{R}^{-1}=\mathbf{T}^{-1}(\mathbf{T}^{\mathbf{T}})^{-1}$, the term in brackets in equation (4.16), may be rewritten as follows:

$$\begin{aligned}&\mathbf{K}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{TK}-\mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}-\mathbf{PBK}=\\&=\mathbf{K}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{TK}-\mathbf{K}^{\mathrm{T}}\left[\mathbf{T}^{\mathrm{T}}\left(\mathbf{T}^{\mathrm{T}}\right)^{-1}\right]\mathbf{B}^{\mathrm{T}}\mathbf{P}-\mathbf{P}^{\mathrm{T}}\mathbf{BK}+\\&+\left(\mathbf{P}^{\mathrm{T}}-\mathbf{P}\right)\mathbf{BR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}=\\&=\mathbf{K}^{\mathbf{T}}\mathbf{T}^{\mathbf{T}}\mathbf{TK}-\mathbf{K}^{\mathbf{T}}\mathbf{T}^{\mathbf{T}}\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}-\mathbf{P}^{\mathbf{T}}\mathbf{B}\left(\mathbf{T}^{-1}\mathbf{T}\right)\mathbf{K}+\\&+\mathbf{P}^{\mathbf{T}}\mathbf{B}\left[\mathbf{T}^{-1}\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\right]\mathbf{B}^{\mathbf{T}}\mathbf{P}-\mathbf{PBR}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}=\\&=\left[\mathbf{K}^{\mathbf{T}}\mathbf{T}^{\mathbf{T}}-\mathbf{P}^{\mathbf{T}}\mathbf{BT}^{-1}\right]\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]-\mathbf{PBR}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}=\\&=\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}}\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]-\mathbf{PBR}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\end{aligned} \quad . \quad (4.17)$$

Thus, equation (4.16) may be derived as:

$$\begin{aligned}&\mathbf{A}^{\mathbf{T}}\mathbf{P}+\mathbf{PA}+\\&+\left[\mathbf{TK}\text{-}\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}}\left[\mathbf{TK}\text{-}\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]-\\&-\mathbf{PBR}^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}+\mathbf{Q}=0\end{aligned} \quad (4.18)$$

Minimizing integral criteria (4.2), in other words, derivation of the optimal state feedback static gain matrix **K**, means mimization of the matrix product of

$$\mathbf{x}^{\mathrm{T}}\left[\mathbf{TK}-\left(\mathbf{T}^{\mathrm{T}}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\right]^{\mathrm{T}}\left[\mathbf{TK}-\left(\mathbf{T}^{\mathrm{T}}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\right]\mathbf{x}\,.(4.19)$$

Since matrix defined by equation is a non-negative one, thus equation (4.18) takes a minimum if

$$\mathbf{TK}=\left(\mathbf{T}^{\mathrm{T}}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}. \quad (4.20)$$

Let us find optimal static feedback gain matrix **K** from equation (4.20), thus we have:

$$\mathbf{K}^{\mathbf{o}}=\mathbf{T}^{-1}\left(\mathbf{T}^{\mathrm{T}}\right)^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}=\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}. \quad (4.21)$$

The optimal control law is as follows below:

$$\mathbf{u}^{\mathbf{o}}(t)=-\mathbf{K}^{\mathbf{o}}\mathbf{x}(\mathrm{t})=-\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Px}(\mathrm{t}). \quad (4.22)$$

For derivation of matrix **P** there is often used method of algebraic Ricatti equation (ARE):

$$\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{PA}-\mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}+\mathbf{Q}=0. \quad (4.23)$$

Solution of the LQR controller synthesis problem consists of following steps: using equation (4.23) positive definite cost matrix **P** must be derived; substituting matrix **P** into equation (4.22), what is optimal control law.

The optimal static feedback gain matrix **K** may be derived using MATLAB supplemented with Control System Toolbox. The built-in files of the proposed software may be used for solution of this problem are *lqr.m*, and *lqr2.m*.

4.2 Preliminary design of the Height Control System of the Quadrotor. Block diagram of the height control system of the UAV may be seen in Figure 5.

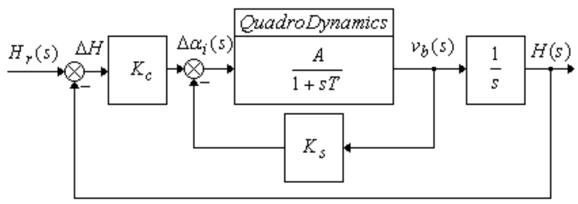

Figure 5. UAV Height Control System.

Using Figure 5. state equations of the UAV height control system of the quadrotor may be derived as:

$$v_b(s)=\frac{A}{1+sT}\Delta\alpha_i(s)\rightarrow \\ \rightarrow \dot{v}_b(t)=-\frac{v_b(t)}{T}+\frac{A}{T}\Delta\alpha_i(t), \qquad (4.24)$$

$$H(s)=\frac{1}{s}v_b(s)\rightarrow \dot{H}(t)=v_b(t), \qquad (4.25)$$

or in matrix form:

$$\dot{\mathbf{x}}(t)=\begin{bmatrix}\dot{v}_b(t)\\ \dot{H}(t)\end{bmatrix}=\begin{bmatrix}-\frac{1}{T} & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_b(t)\\ H(t)\end{bmatrix}+\begin{bmatrix}A/T\\ 0\end{bmatrix}\Delta\alpha_i(t) \qquad (4.26)$$

Results of the uncontrolled UAV transient response analysis may be seen in Figure 6.

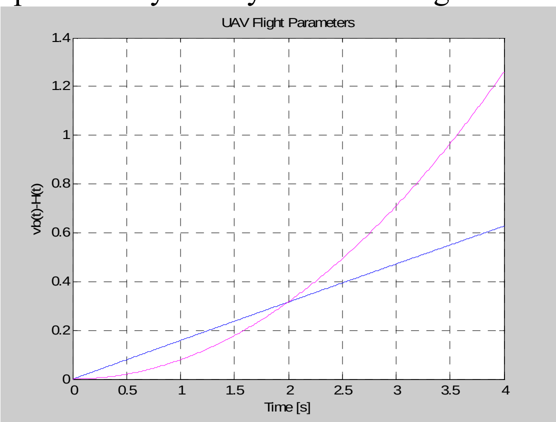

Figure 6. Uncontrolled UAV Step Responses
Vertical Speed Altitude

Control law of the closed loop control system may be found using Figure 5, and it is as follows below:

$$\mathbf{u}(t)=\Delta\alpha_i(t)=-H(t)K_c-v_b(t)K_s=-\mathbf{Kx}, \qquad (4.27)$$

where: $\mathbf{x}=[v_b \quad H]^T$ is the state vector, $\mathbf{K}=[K_c \quad K_s]$ is the static feedback gain matrix.

Let find the optimal static feedback gain matrix, i.e. the optimal control law. For the first set of weighting matrices choose them by rule of unit weights, thus, we have:

$$\mathbf{Q}_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}; r_1=1. \qquad (4.28)$$

In this case the static gain is as follows [3]:

$$\mathbf{K}_1=[K_c \quad K_s]=[3{,}6449 \quad 1]. \qquad (4.29)$$

Results of the computer simulation of the closed loop control system for $H_r(t)=1(t)$ may be seen in Figure 7 [3, 7].

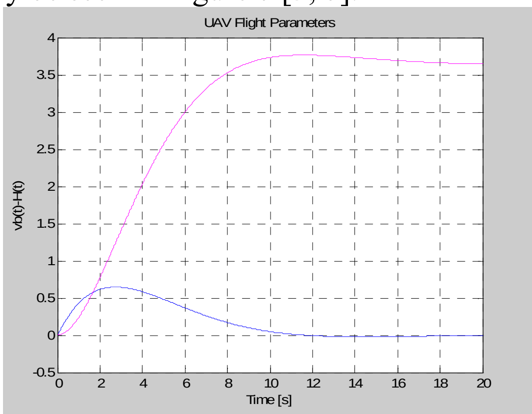

Figure 7. Controlled UAV Step Responses
Vertical Speed Altitude

Dynamic performances of the closed loop height control system designed by weights of (4.28) are as follows:

Eigenvalues	Damping ratio, ξ	Frequencies, [rad/s]
$-0{,}293\pm 0{,}27i$	0,735	0,399

From Figure 7 it may be seen that steady-state value of the height of the flight is $H(\infty)\approx 3{,}7m$, i.e. the reference signal yields to larger output from the system, and the closed loop control system dynamic performances are do not match those dynamic performances defined in [14]. However it is worth to mention that due to lack of complex set of dynamic performances for UAV automatic flight control systems, the standard [14] what is for aircraft piloted by human, was used instead. Let us change weighting matrices defined by equation (4.28) heuristically, to be as follows:

$$\mathbf{Q}_2=\begin{bmatrix}0{,}97 & 0\\ 0 & 1\end{bmatrix}; r_2=0{,}000005. \qquad (4.30)$$

Using weights (4.30) the static feedback gain matrix was found to be [3, 8]:

$$\mathbf{K}_2 = [K_c \quad K_s] = [446,7565 \quad 447,2136] . (4.31)$$

Results of the computer simulation of the closed loop control system for given weights of (4.30), and for given step function of $H_r(t) = 1(t)$, may be seen in Figure 8 [3, 7].

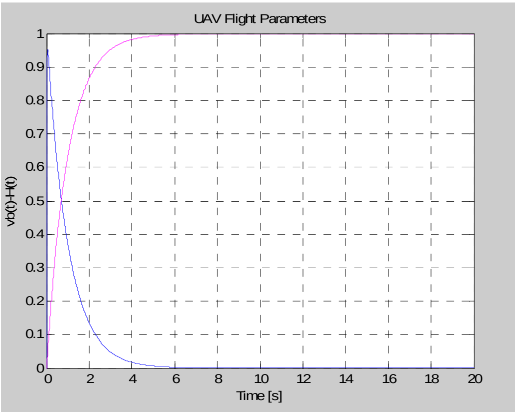

Figure 8. Controlled UAV Step Responses
Vertical Speed Altitude

Dynamic performances of the UAV closed loop height control system designed by weights of (4.30) are as follows:

Eigenvalues	Damping ratio, ξ	Frequencies, [rad/s]
-70	1	70
$-1{,}02$	1	1,02

From Figure 8. it may be seen that steady-state value of the height of the flight is $H(\infty) = 1m$.

In other words, the unit value reference input is followed with no static error, and the closed loop dynamic performances are those defined for aircraft as given in [14].

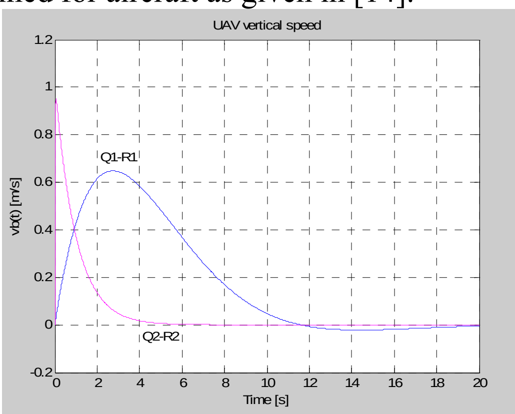

Figure 9-a. Controlled UAV Step Responses
Q1-R1 Q2-R2.

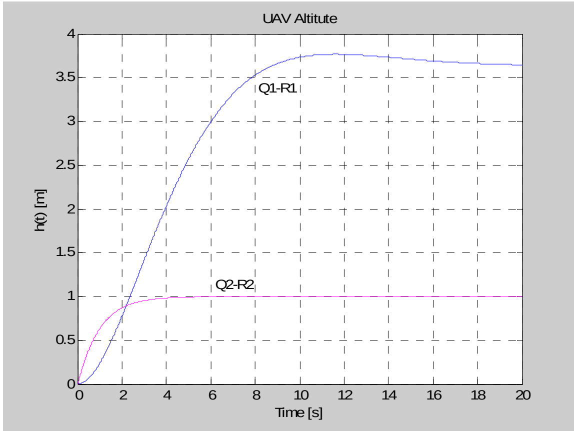

Figure 9-b. Controlled UAV Step Responses
Q1-R1 Q2-R2.

Results of the computer simulation of two designed systems for weights of (4.28), and (4.30), may be seen in Figure 9. From Figure 9. it is easily may be derived that heuristically set weighting matrices can derive namely that optimal control law, what will be able to provide dynamic performances of the closed loop altitude control system of the UAV [3, 7].

5. CONCLUSIONS

This paper deals with optimal control system design. The method propagated here is the LQR one, which is widely applied as preliminary design method for controller synthesis of UAVs.

The optimal control law synthesis is executed using heuristic setting of the weighting matrices in integral performance index. Dynamic performances were considered for those defined for piloted aircraft.

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