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ABOUT SMOOTHING FUNCTIONS USED IN SPH METHOD

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ABSTRACT. A fundamental issue of Smoothed Particle Hydrodynamics (SPH) theory is the smoothing function, often called smoothing kernel function, or smoothing kernel or simply kernel.
The kernel function determines the pattern of the function approximation, determines the consistency, the accuracy of the results. So, a maximum attention must be paid to the smoothing function, because by a right choosing (when this is possible) we can improve the results.
This paper presents some theoretical consideration upon smoothing functions, some requirements for these and how this issue is implemented and available in Ls-Dyna program. Also, some examples are presented, which represent fundamentals of the final conclusions.

KEYWORDS: Smoothed Particle Hydrodynamics, kernel function, smoothing length.

1. INTRODUCTION

Smoothed particle hydrodynamics (SPH) is a meshfree Lagrangian particle method having a short history comparatively with finite difference method (FDM) or finite element method (FEM). Its begining can be found in 1977, when it was used to solve astrophysical problems in three-dimensional open space.

Nowadays the SPH method is being used in many engineering fields. Between these, the numerical modeling of fluid flows is one of a great succes, but not many years ago, SPH was also used in applied mechanics.

Many special softwares were created and others well known powerfull numerical programs implemented this new method, SPH.

In our country this method is less used despite its advantages in solving of the problems involving large deformation, free surface etc. Our paper offers some specific information for an easier understanding and even using of SPH method.

2. INTEGRAL REPRESENTATION OF A FUNCTION

The theoretical fundamentals of the SPH method can be approched in two steps. The first is the integral representation or kernel approximation of the field functions.

The second one is the approximation of particle parameters (mass, velocity, etc.).

Integral representation of a function $f(x)$, used in the SPH method starts from the following identity:

$$f(x) = \int_{\Omega} f(x')\delta(x - x')dx' \quad (1)$$

where f is a function of a position vector x, which can be an one-, two- or three-dimensional one; $\delta(x - x')$ is a Dirac function, having the properties:

$$\delta(x-x')=\begin{cases}1\rightarrow x=x'\\0\rightarrow x\neq x'\end{cases}$$
(2)

In equation (1), Ω is the function domain, which can be a volume, that contains the x, and where $f(x)$ is defined and continuous.

By replacing the Dirac function with a smoothing function $W(x-x',h)$ the integral representation of $f(x)$ becomes:

$$f(x)=\int_{\Omega} f(x')W(x-x',h)dx'$$
(3)

where W is the smoothing kernel function, or smoothing function, or kernel function.

The parameter h, of the smoothing function W, is the smoothing length, by which the influence area of the smoothing function W is defined (Figure 1-a and 1-b).

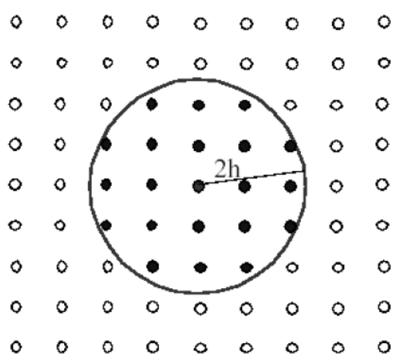


Fig. 1-a Support domain of W

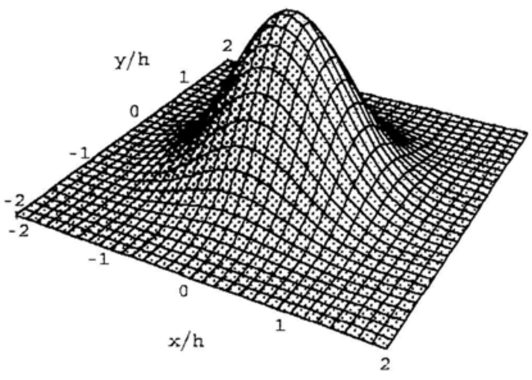


Fig. 1-b Graphical representation of 2D-Kernel function

As long as Dirac delta function is used, the integral representation, described by equation (1), is an exact (rigorous) one, but using the smoothing function W instead of Dirac function, the integral representation can only be an approximation. This is the reason for the name of kernel approximation. Using the angle bracket $\langle\ \rangle$ this aspect is underlined and the equation (3) can be rewritten as:

$$\langle f(x)\rangle=\int_{\Omega} f(x')W(x-x',h)dx'$$
(4)

The smoothing function W is usually chosen to be an even one, which has to satisfy some conditions.

The first condition, named normalization condition or unity condition is:

$$\int_{\Omega} W(x-x',h)dx'=1$$
(5)

The second condition is the Delta function property and it occures when the smoothing length approaches zero:

$$\lim_{h\rightarrow 0} W(x-x',h)=\delta(x-x')$$
(6)

The third condition is the compact condition, expressed by:

$$W(x-x',h)=0 \qquad \text{when} \qquad |x-x'|>kh$$
(7)

where k is a constant related to the smoothing function for point at x, defining the effective non-zero area of the smoothing function as the figure 2 shows.

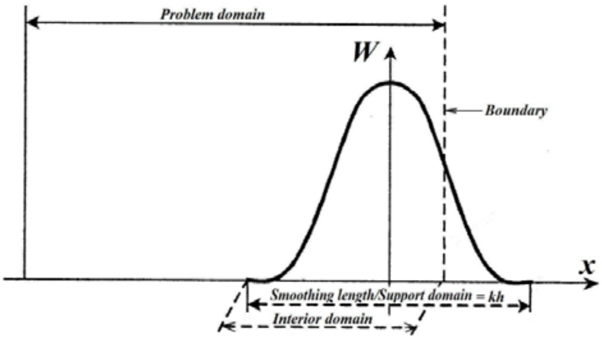


Fig. 2 Smoothing length

As the particle approximation is concerned, the continuous integral aproximation (4) can be converted to a summation of discretized forms, over all particles belonging to the support domain.

Changing the infinitesimal volume dx' with the finite volume of the particle ΔV_j, the mass of the particles m_j can be written,

$$m_j = \Delta V_j \rho_j \quad (8)$$

and finally, relation (3) becomes:

$$f(x) = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) W(x - x_j, h) \quad (9)$$

The particle approximation of a parameter described by a function, for particle i can be expressed by,

$$\langle f(x_i) \rangle = \sum_{j=1}^{N} \frac{m_j}{\rho_j} f(x_j) W_{ij} \quad (10)$$

where $W_{ij} = W(x_i - x_j, h)$. (11)

3. PROPERTIES OF THE SMOOTHING FUNCTIONS

In the SPH literature, various requirements of the smoothing function are debated.

The most important of them (the first 7th) are presented below.

- the smoothing function has to be ***normalized*** over its support:

$$\int_{\Omega} W(x - x', h) dx' = 1 \quad (12)$$

- the smoothing function has to be ***compactly supported:***

$$W(x - x', h) = 0 \text{ for } |x - x'| > kh \quad (13)$$

- the smoothing function has to be ***positive*** for any point at x' within the support domain:

$$W(x - x', h) \geq 0 \quad (14)$$

- the smoothing function value has to be ***monotonically decreasing*** with the increase of the distance away from the particle.
- the smoothing function value has to satisfy the ***Dirac delta function*** condition as the smoothing length approaches to zero:

$$\lim_{h \to 0} W(x - x', h) = \delta(x - x') \quad (15)$$

- the smoothing function value has to be an ***even function*** (symetric).
- the smoothing function value has to be sufficiently smooth (smoothness).

4. SMOOTHING FUNCTIONS

Published literature presents different smoothing function (also called smoothing kernel function, smoothing kernel, or kernel).

Theoretically, any function having the properties presented above, can be employed as SPH smoothing function. First time, Lucy (1977) used the following bell-shaped function as the smoothing function:

$$W(s,h) = \frac{\alpha}{h^n} \begin{cases} (1+3s)(1-s^2)^3 \leftarrow s \le 1 \\ 0 \leftarrow s > 1 \end{cases} \quad (16)$$

where α is $\frac{5}{4}$, $\frac{5}{\pi}$ or $\frac{105}{16\pi}$, n is a number representing the space dimension, $s = \frac{|x - x'|}{h}$ or $s == \frac{r}{h}$, r being the distance between two points (particles).

The graphical representation of this smoothing function and its derivatives (first and second) can be seen in the figure 3.

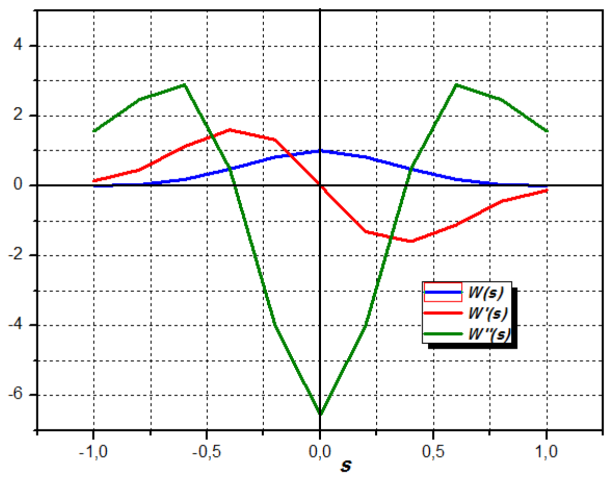


Fig. 3 Smoothing function and its derivates, used by Lucy in 1977

Monaghan in 1992 and Gingold and Monaghan in 1977 assumed the smoothing function to be a Gaussian, expressed by:

$$W(s,h) = \frac{\alpha}{h^n} e^{-s^2} \quad (17)$$

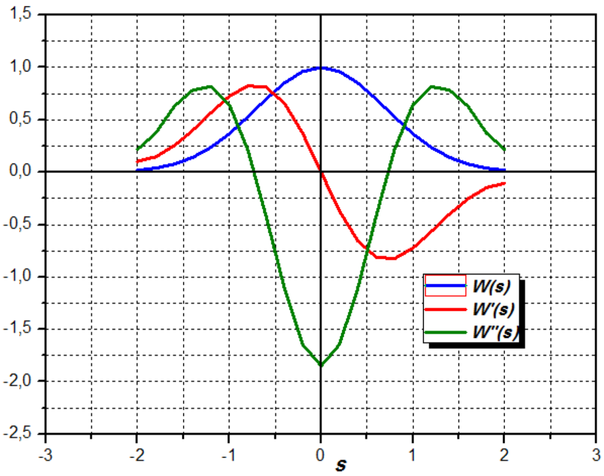


Fig. 4 Smoothing function and its derivates, used in 1977 and 1992 (Ginglod and Monaghan)

Many notations used in relation (16) are the same used in previous type of kernel.

The notation α has the following expression: $\frac{1}{\pi^{0.5}}$, $\frac{1}{\pi}$ or $\frac{1}{\pi^{1.5}}$ in function of the space dimension (1D, 2D or 3D).

The graphical representation of this smoothing function and its derivatives (first and second) can be seen in the figure 4.

Monaghan and Lattanzio, in 1985, used a smoothing function based on the cubic spline function, named B-spline function:

$$W(s,h) = \frac{\alpha}{h^n} \begin{cases} \frac{2}{3} - s^2 + \frac{1}{2}s^3 \leftarrow 0 \le s < 1 \\ \frac{1}{6}(2-s)^3 \leftarrow 1 \le s < 2 \\ 0 \leftarrow s \ge 2 \end{cases} \quad (18)$$

The constant α has the values 1, $\frac{15}{7\pi}$ or $\frac{3}{2\pi}$ in function of the space dimension (1D, 2D or 3D). This type of smoothing function, so far, it is the most widely used, specially in dedicated software.

The graphical representation of this smoothing function and its derivatives (first and second) can be seen in the figure 5.

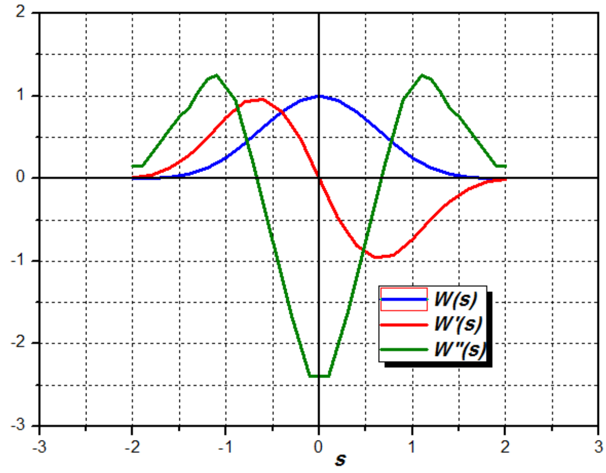


Fig. 5 B-spline smoothing function and its derivates

Spline functions of higher order (quartic and quintic) can be more closely in approximating the Gaussian and they are more stable.

The expression of a quintic spline function (Morris, 1996) is:

$$W(s,h)=\frac{\alpha}{h^n}\begin{cases}(3-s)^5-6(2-s)^5+\\ +15(1-s)^5 \leftarrow 0\le s<1\\ (3-s)^5-6(2-s)^5\\ for \rightarrow 1\le s<2\\ (3-s)^5 \leftarrow 2\le s<3\\ 0\leftarrow s>3\end{cases} \tag{19}$$

The graphical representation of this smoothing function and its derivatives (first and second) can be seen in the figure 6.

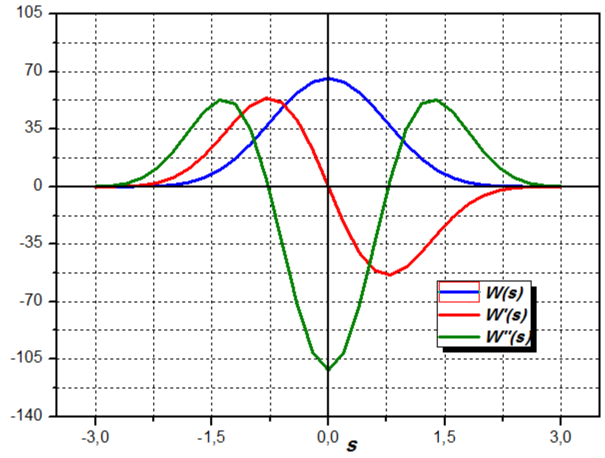


Fig. 6 Quintic spline smoothing function and its derivates

In 1996, Johnson et al. used a quadratic smoothing function to simulate the high velocity impact problem.

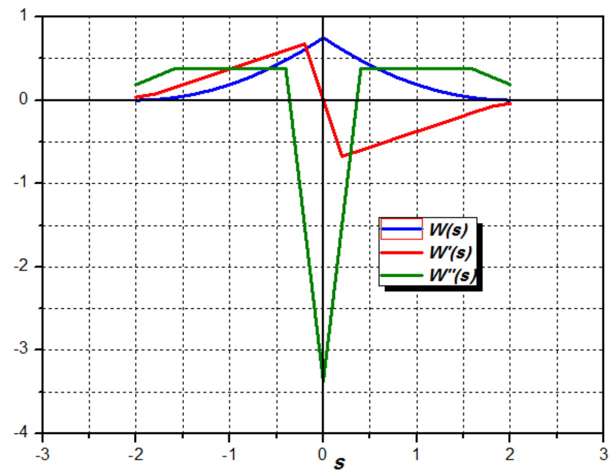


Fig. 7 Quadratic spline smoothing function and its derivates

The graphical representation of this smoothing function and its derivatives (first and second) can be seen in the figure 7.

The expression of the Johnson smoothing function is:

$$W(s,h)=\frac{\alpha}{h^n}\left(\frac{3}{16}s^2-\frac{3}{4}s+\frac{3}{4}\right) \qquad (20)$$

for s being between zero and two ($0 \le s \le 2$).

5. SMOOTHING FUNCTIONS IN LS-DYNA PROGRAM

One of the most powerfull program for simulation of the dynamic problems, which has the SPH method implemented, is Ls-Dyna.

This program uses a cubic B-spline kernel function, described above.

The user can make a choosing regarding to the the particle approximation, having the following options, by FORM parameter (CONTROL_SPH): default formulation (0), renormalization approximation (1), symmetric formulation (2), symmetric renormalized approximation (3), tensor formulation (4), fluid particle approximation (5), or fluid particle with renormalization approximation (6). These options can be made

Others options can be made regarding to the computation or not of the particle approximation between two different SPH parts and regarding to the time integration type for the smoothing length h:

$$\frac{d}{dt}\big(h(t)\big)=\frac{1}{d}h(t)div(v)\,, \qquad (21)$$

or,

$$\frac{d}{dt}\big(h(t)\big)=\frac{1}{d}h(t)\big(div(v)\big)^{1/3} \qquad (22)$$

The smoothing length h, can be calculated by the program, just the calculus begining, if this is permited to be variable during computing simulation, or can has a defined values, established by the user (using parameters CSLH, HMIN and HMAX, of SECTION_SPH).

6. NUMERICAL TESTS

An well known experimental test, Taylor test, is presented by numerical simulations, using Ls-Dyna program. The test consist in the impact between a cilynder with a rigid wall.

A solid cylinder, having a velocity of 200 m/s, with radius of 5 mm and the length of 50 mm, made of 1018 steel was considered.

Two numerical models studied the impact between this metal rod with a rigid wall: FEM and SPH models. Finite element model was made using 2993 nodes and 2560 elements (element size being 1.250x1.077x1.077 mm) and can be seen in the figure 8.

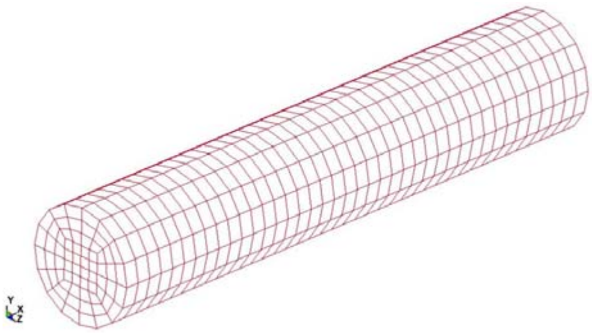
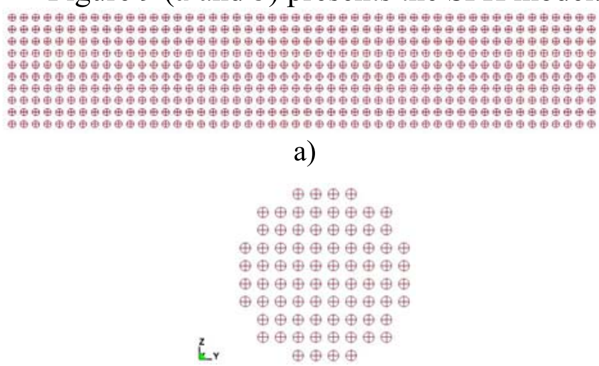


Fig. 8 Finite element model

SPH model consisted in 4000 particles (equal distance between particles 1.00 mm).

Figure 9 (*a* and *b*) presents the SPH model.



a)

b)

Fig. 9 SPH model of the bar

For both models, the fundamental measure units were: for length millimeter [*mm*], for time second [*s*] and for force Newton [*N*].

Analysis time was established at 0.003 seconds, for the stress and displacement field analysis. in a period after the impact, when the velocity changed its sign.

The study of material behavior was based on plastic-kinematic material model.

In the Figure 10, deformed shape and UX-displacement field are presented, for FE and

SPH modeling, for the time of 6e-5 s.

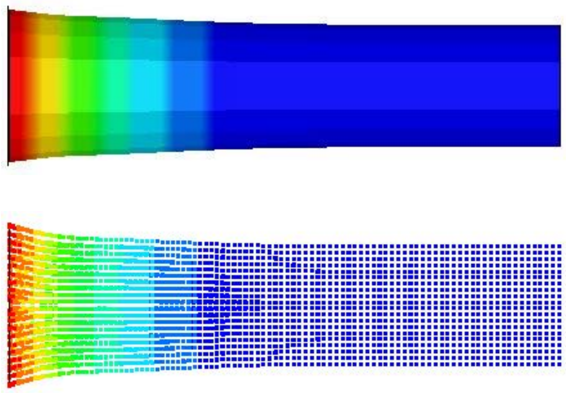


Fig. 10 UX-displacement field

Table 1. Impact effects upon the bar

Models	Bar Head		Bar Tail	
	UX_{max} mm	VX_{max} mm/s	UX_{max} mm	VX_{max} mm/s
FEM	43.390	26557	38.404	28720
SPH	43.475	27295	38.172	26891
	Er. 0,2%	*Er. 2,77%*	*Er. -0.6%*	*Er. -6,4%*

Table 1 presents some of the results for the default values of SPH using. In the table 2, the same results are presented for different values of the parameter FORM.

Table 2. The influence of the kernel

	Bar Head		Bar Tail	
	UX_{max} mm	VX_{max} mm/s	UX_{max} mm	VX_{max} mm/s
FORM=1	42.592	25176	37.866	27575
Er. [%]	*-1.84*	*-5.20*	*-1.40*	*-3.98*
FORM=2	43.556	25371	38.245	27010
Er. [%]	*0.38*	*-4.46*	*-0.41*	*-5.95*
FORM=3	42.055	24571	37.302	26559
Er. [%]	*-3.10*	*-7.47*	*-2.87*	*-7.52*
FORM=5	43.545	25384	38.235	26997
Er. [%]	*0.36*	*-4.41*	*-0.44*	*-5.99*
FORM=6	42.272	25055	37.546	27687
Er. [%]	*-2.57*	*-5.65*	*-2.23*	*-3.59*

7. CONCLUSIONS

Using of different smoothing functions leads us to different results, which could be far enough from the reality.

The choosing of the smoothing function and the options referring to this as well has a great importance for the calculus results.

The default values offered by Ls-Dyna program leads us to the best results, for classical conditions regarding to the material and its loading.

The choosing possibility of different smoothing functions (or different versions of these) must be studied, because this aspect allow us to do some numerical calibration of the SPH method, as the results to be the best.

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