"HENRI COANDA"
AIR FORCE ACADEMY
ROMANIA

"GENERAL M.R. STEFANIK"
ARMED FORCES ACADEMY
SLOVAK REPUBLIC

INTERNATIONAL CONFERENCE of SCIENTIFIC PAPER
AFASES 2015
Brasov, 28-30 May 2015

AIRPLANES OR RACKETS FLIGHT STABILIZATION OPTIMAL CONTROL IN CASE OF PITCH PERTURBATIONS

Mircea Lupu*, Cristian-George Constantinescu, Gheorghe Radu****

*Corresponding member of Romanian Scientists Academy, Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Romsnia, **Faculty of Aeronautical Management, "Henri Coandă" Air Force Academy, Brasov, Romania

Abstract: *In this paper the study of horizontal flight stabilization by using an automate system to control the pitch perturbations will be approached. This optimal control method is based on the extreme principle of Pontreaghin, finding the control function via the minimum transfer time from the initial (disrupted) position in the final position (target). The optimal control* U^* *is determined, as well as the optimal trajectories which solve the optimal control problem (O.C.P.), by using a relay-type regulator with rapid action to stabilize this controllable system, suitable for aircrafts of rockets equipped with autopilot.*

Keywords: *optimal control, control function, minimum transfer time.*
***MSC2010*:** 76B07, 49Q10, 35J25, 35F15, 45E05.

1. INTRODUCTION

In case of horizontal flight, pitch (longitudinal) perturbation may occur, pitch angle ψ varying in this way. Considering an axis system Oxyz, the origin O being the CoG (mass center) of the aircraft, with Ox the horizontal flight axis, Oz as vertical axis and Oy as lateral axis, the pitch perturbation are characterized by the pitch angle ψ, (the rotation angle around Oy), lateral perturbations characterized by the rolling angle θ (the rotation angle around Ox) and yaw angle φ (the rotation angle around Oz). These angles, together with their corresponding moments, are shown in figure 1.

In previous studies, [1], [2] it was presented the automate control and absolute stabilization for a linearised dynamic system by using two (equivalent) methods:

- the Lurie method, finding the Liapunov function [4-6, 9];
- the frequencial method V. M. Popov, using the transfer function, [8].

In this paper the study of horizontal flight stabilization by using an automate system to control the pitch perturbations will be approached. This optimal control method is based on the extreme principle of Pontreaguine, [7], finding the control function via the minimum transfer time: $\min(t_1 - t_0) = t^*$, from the initial (disrupted) position: $P_0(X^0; t_0)$ in the final position (target): $O(X^1; t_1)$.

So, the optimal control U^* is determined, as well as the optimal trajectories $X(X^0; X^1; t_0; t_1; t)$ which solve the optimal control problem (O.C.P.), by using a relay-type (on-off) regulator with rapid action to stabilize this controllable system, suitable for aircrafts of rockets equipped with autopilot.

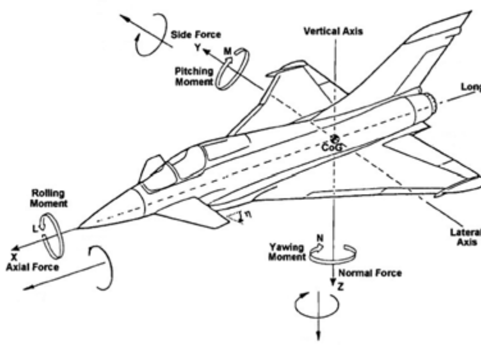

Fig. 1: The axial moments and angles of an airplane

In the previous study it was determined the Liapunov function $V \geq 0$, with $\dot{V} \leq 0$, ensuring the absolute global stabilization towards the point O in the phase space $(x_1;x_2;x_3) \leftrightarrow (z_1;z_2;z_3)$ and finding the parameters r, C_1, C_2 and the control function $\sigma = C' z - r\varphi$:

$$V = -\frac{z_1^2}{2r_1} - \frac{z_2^2}{2r_2} + \frac{a}{2} z_3^2 + \int_0^{\sigma} f(\sigma) d\sigma$$

where $a = \frac{C_1}{b_3 \lambda_1 \lambda_2} > 0$, and φ is an arbitrary non-linear control function; $\varphi(\sigma)$ respects the sector conditions. The system is fully controllable because the rank $R = (b; Ab; A^2 b) = 3$

2. MATHEMATICAL DETERMINATION

In the following section of the paper it will be presented the optimal control in the space $Oz_1z_2z_3$ for the diagonal system (see [3]), as: $\dot{z} = rz + bu \qquad \varphi = u$, detailed in the form:

$$\begin{cases} \dot{z}_1 = r_1 z_1 + bu \\ \dot{z}_2 = r_2 z_2 + bu \\ \dot{z}_3 = bu \end{cases} \tag{1}$$

where:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}; u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}; u_1 = u_2 = u_3 := u;$$

$$r_2 < r_1 < 0; r_3 = 0; b > 0$$

$$\left\{ \dot{z} = \frac{\partial H}{\partial \lambda}; \dot{\lambda} = -\frac{\partial H}{\partial z} \right\} \tag{2}$$

$$H = 1 + \lambda_1 \dot{z}_1 + \lambda_2 \dot{z}_2 + \lambda_3 \dot{z}_3 \tag{3}$$

$$H = \begin{Bmatrix} 1 + \{\lambda_1 r_1 z_1 + \lambda_2 r_2 z_2 + \lambda_3 r_3 z_3\} + \\ + bu\{\lambda_1 + \lambda_2 + \lambda_3\} \end{Bmatrix} \tag{4}$$

$$H \geq 0 \Rightarrow H_{min} = H(t^*, u^*, z^*, \lambda^*) = 0$$

$$|u| \leq 1 \Rightarrow u^* = \pm 1$$

As a function of u, the Hamiltonian H is a linear dependence:

$$H = H(u) = Bu + C$$

$$|u| \leq 1 \Rightarrow H_{min} = H(u^*) = 0$$

If $u = u^* = -1$, it results that H is an increasing function, and if $u = u^* = 1$, it results that H is a decreasing function, as shown in figure 2.

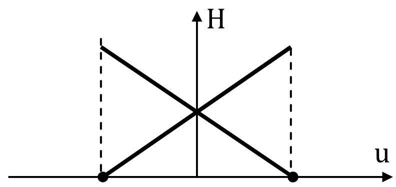

Fig. 2: The dependence H = H(u)

$$u^* = -\text{sign}(\lambda_1 + \lambda_2 + \lambda_3); b > 0$$

$$F = \lambda_1 + \lambda_2 + \lambda_3$$

$$\dot{\lambda} = -\frac{\partial H}{\partial z} \Rightarrow \begin{cases} \dot{\lambda}_1 = -\lambda_1 r_1 \\ \dot{\lambda}_2 = -\lambda_2 r_2 \\ \dot{\lambda}_3 = 0 \end{cases}; \frac{d\lambda}{\lambda} = -r \tag{5}$$

$$\begin{cases} \lambda_1 = C_1 e^{-r_1 t} \\ \lambda_2 = C_2 e^{-r_2 t} \\ \lambda_3 = C_3 \end{cases}; \lambda_i|_{t=0} = C_i \tag{5'}$$

$$F = F(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t} + C_3$$

$$C_{1,2,3} \in \mathrm{R}, C_1 C_2 C_3 \neq 0$$

$$u^* = -\text{sign}(F(t))$$

Remark:

The regulator is a relay-type one, so $u^* = \pm 1$. For $t \in [0; t^*]$ the sign of F may be constant, so u^* keeps its value (± 1) along the whole above mentioned time interval. It means that in this case no relay commutation will occur. This situation is possible if C_1, C_2 and

C_3 are simultaneously either positive or negative.

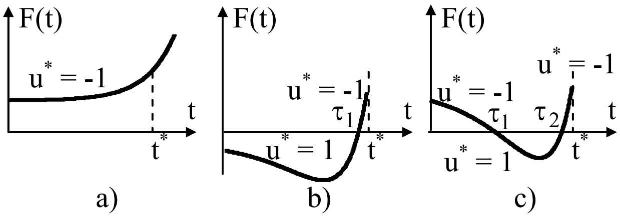

Fig. 3: Relay commutation
a) no commutation;
b) one commutation;
c) two commutations.

By varying the signs of C_1, C_2 and C_3, the function F(t) may have one or maximum two sign changes for $t \in \left[0;t^*\right]$, so u^* will change its value too, accordingly to these. So one or maximum two relay commutations may occur in the time interval.

In figure 3 are presented 3 examples of the commutation function F(t).

In this case the Hamiltonian $H\left(t, u^*, z, \lambda\right)$ is computed by splitting the time interval:

- $t \in [0;\tau_1] \cup \left[\tau_1;t^*\right]$ if F(t) has a single sign change (in the moment $t = \tau_1$), as shown in figure 3b;
- $t \in [0;\tau_1] \cup [\tau_1;\tau_2] \cup \left[\tau_2;t^*\right]$ if F(t) has a double sign change (in the moments $t = \tau_1$ and $t = \tau_2$), as shown in figure 3c.

Trajectories $z_i = z_i\left(t, u^*\right)$ must be computed in accordance to the above mentioned.

In conclusion, the control possibilities depend on analysis of relay commutation, computing procedures for F(t) being necessary. As shown in figure 3, a such procedure must be able to solve the equation $F(t) = 0$, and to assign the correct values to the commutation variable: $u^* = -\mathrm{sign}(F(t))$.

The compatibility of choosing C_1, C_2, C_3 and initial conditions (the starting position of the flying object) with the trajectory equations will be conditioned by positive values of time $(t \geq 0)$ and $H\left(t^*, u^*, z^*, \lambda^*\right) = 0$.

2.1. Un-commutated Trajectory. It results from (1) the trajectory equations:

$$\begin{cases} z_1 = \dfrac{1}{r_1}\left[\left(\alpha_1 r_1 + bu^*\right)e^{r_1 t} - bu^*\right] \\ z_2 = \dfrac{1}{r_2}\left[\left(\alpha_2 r_2 + bu^*\right)e^{r_2 t} - bu^*\right] \\ z_3 = bu^* t + \alpha_3 \end{cases} \qquad (7)$$

where $A_0(\alpha_1, \alpha_2; \alpha_3)$ is the initial position.

By eliminating the time between the 3 above equations: $t = \dfrac{z_3 - \alpha_3}{bu^*}$, the trajectory results as the intersection of two surfaces, starting in the moment $t = 0$ from A_0:

$$\begin{cases} z_1 = \dfrac{1}{r_1}\left[\left(\alpha_1 r_1 + bu^*\right)e^{r_1 \frac{z_3 - \alpha_3}{bu^*}} - bu^*\right] \\ z_2 = \dfrac{1}{r_2}\left[\left(\alpha_2 r_2 + bu^*\right)e^{r_2 \frac{z_3 - \alpha_3}{bu^*}} - bu^*\right] \end{cases} \qquad (8)$$

It may be noticed that the surfaces $z_1 = z_1(z_3)$ and $z_2 = z_2(z_3)$ are crossing the planes (z_1Oz_3) and (z_2Oz_3) respectively.

The goal for this trajectory $(z = z(t;\alpha))$ is to reach the target in origin: $z_i\left(t^*\right) = 0, i = \overline{1,3}$.

From $z_3\left(t^*\right) = 0$ it results:

$$t^* = -\frac{\alpha_3}{bu^*} \qquad (9)$$

As time must be positive, the following restriction is obvious:

$$\frac{\alpha_3}{bu^*} < 0 \xrightarrow[b>0]{} \begin{cases} \alpha_3 > 0; u^* = -1 \\ \text{or} \\ \alpha_3 < 0; u^* = 1 \end{cases}$$

Similar, z_1 and z_2 from (8) must be null in the same time with z_3.

$$\begin{cases} z_1\big|_{z_3=0} \Rightarrow \alpha_1 = \dfrac{bu^*}{r_1}\left(1 - e^{-\frac{r_1\alpha_3}{bu^*}}\right) \\ z_2\big|_{z_3=0} \Rightarrow \alpha_2 = \dfrac{bu^*}{r_2}\left(1 - e^{-\frac{r_2\alpha_3}{bu^*}}\right) \end{cases} \quad (10)$$

In conclusion, if the initial coordinates α_1 and α_2 respect the constraints (10), and with an arbitrary value of α_3, the ending point of the trajectory will be the origin O(0; 0; 0).

The trajectory may be plotted, either by using its parametric equations (7) or as intersection of two surfaces (8).

The mathematical determination of the parametric link between C_1, C_2, C_3 having the same sign (in the non-commutated case) is worked out by recalling the condition:

$$H\left(t^*, u^*, z^*, \lambda^*\right) := H_{min} = 0$$

The optimal trajectories A_0O are $\Gamma = \left\{\Gamma^+\right\} \cup \left\{\Gamma^-\right\}$, corresponding to the values $u = 1$ and $u = -1$, respectively.

Two possible trajectories are shown in figure 4.

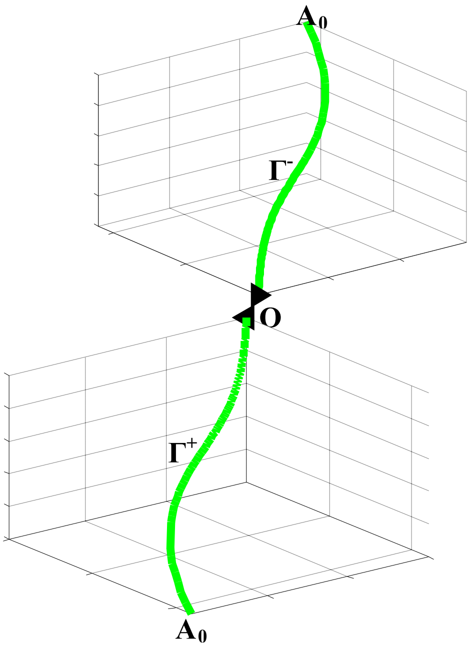

Fig. 4: Un-commutated trajectories

Finally, it comes out that:

$$H\left(u^* = \pm 1\right) = 0 \Leftrightarrow$$
$$\Leftrightarrow 1 \mp b\left(C_1 e^{-r_1 t^*} + C_2 e^{-r_2 t^*} + C_3\right) = 0 \quad (11)$$

where $\mp$ must be the opposite sign of sign(F), defined in (5). So, it results a relationship involving C_1, C_2, C_3, that has to be respected. The conclusion is that only two of these three constants are arbitrary.

A non-commutated case that may be effectively approached is $C_3 = 0$, $C_1, C_2 > 0$, and linked by (11).

2.2. One-commutation trajectory.

In this case the relay switches at a moment $\tau^* < t^*$. The case $C_3 = 0, C_1 < 0$, $C_2 > 0$ is studied further.

Considering $r_2 < r_1 < 0$, and denoting $r_2 = -n$, $r_1 = -m$, with $0 < m < n$, F becomes:

$$F = C_1 e^{mt} + C_2 e^{nt} = 0 \text{ for } 0 < \tau^* < t^*,$$

where τ^* is the switching moment, and it may be computed as it follows:

$$t^* = \frac{1}{m-n}\ln\left(-\frac{C_2}{C_1}\right) = \frac{1}{r_2 - r_1}\ln\left(-\frac{C_2}{C_1}\right)$$

It's obvious that $t^* > 0 \Leftrightarrow \left|\dfrac{C_2}{C_1}\right| < 1$.

In this case, the starting point is $B(\beta_1;\beta_2;\beta_3)$, at the moment $t = 0$, with the command $u^* = -\text{sign}(C_1 + C_2) \underset{|C_1|>|C_2|}{\overset{C_1<0;C_2>0}{=}} 1$.

This trajectory - Γ^+ - will commutate $\left(u^* = -1\right)$ at the moment $t = t^*$, in the point $Q(\gamma_1;\gamma_2;\gamma_3)$. From this moment on, the trajectory will go on with the curve Γ^-, till the target $O(0;0;0)$. The point $Q\left(\gamma;\tau^*\right)$ is similar to A_0, the starting point in the previous (un-commutated) case, but with the initial moment t_0 changed from $t_0 = 0$ to $t_0 = \tau^*$ and with recalculated the final time, t^*.

So, the minimal final time will be $t^* = t_1^* + \tau^*$, for the trajectory described above: $\Gamma = \left\{\Gamma^+\right\} \cup \left\{\Gamma^-\right\} = [BQ] \cup [QO]$.

The phenomena are the same if $C_3 = 0, C_1 > 0, C_2 < 0$, when in the initial moment $t_0 = 0$ the command becomes $u^* = -1$. The trajectory starts from a point B', switches to $u^* = 1$ in the $t = \tau^*$ moment and will continue till the final point, O: $\Gamma = \{\Gamma^-\} \cup \{\Gamma^+\} = [B'Q'] \cup [Q'O]$.

The trajectory Γ must end in O (a fix point) respecting the compatibility conditions. In the following lines, the appropriate mathematical approach will be described.

The solutions for the first curve, BQ, with $t_0 = 0$, are of type (7):

$$\begin{cases} z_i = \dfrac{1}{r_i}\left[\left(\beta_i r_i + bu^*\right)e^{r_i t} - bu^*\right], i = 1;2 \\ z_3 = bu^* t + \beta_3 \end{cases} \tag{7'}$$

or of type (8):

$$\begin{cases} z_i = \dfrac{1}{r_i}\left[\left(\beta_i r_i + bu^*\right)e^{r_i \frac{z_3 - \beta_3}{bu^*}} - bu^*\right] \\ i = 1;2 \end{cases} \tag{8'}$$

The coordinates of the point Q are obtained from (7'), at the moment $t = \tau^*$:

$$\begin{cases} \gamma_i = \dfrac{1}{r_i}\left[\left(\beta_i r_i + bu^*\right)e^{r_i \tau^*} - bu^*\right] \\ \gamma_3 = bu^* \tau^* + \beta_3 \end{cases} \tag{7''}$$

The solutions for the second curve, QO, with $t_0 = \tau^*$ and $u^* \to -u^*$ corresponding to the first curve, BQ, are of type (7) too:

$$\begin{cases} z_i = \dfrac{1}{r_i}\left[\left(\gamma_i r_i - bu^*\right)e^{r_i\left(t - \tau^*\right)} + bu^*\right] \\ z_3 = -bu^*\left(t - \tau^*\right) + \gamma_3 ; t \geq \tau^* > 0 \end{cases}$$

When the trajectory reaches the origin, its parametric equations must nullify:

$$z_i\left(t^*\right) = 0, i = \overline{1;3}$$

$$z_3\left(t^*\right) = 0 \Rightarrow t^* = \tau^* + \frac{\gamma_3}{bu^*}$$

The coordinates γ_i must respect the compatibility constraints:

$$z_i\left(t_f\right) = 0, i = 1;2 \Rightarrow$$

$$\Rightarrow \left(\gamma_i r_i - bu^*\right)e^{r_i\left(t_f - \tau^*\right)} + bu^* = 0$$

It results:

$$\gamma_i = \frac{bu^*\left(e^{r_i\left(t_f - \tau^*\right)} - 1\right)}{r_i e^{r_i\left(t_f - \tau^*\right)}} \tag{12}$$

Similar compatibility constraints must be respected by the coordinates β, which has to be computed from (7''), replacing γ_i with the one computed with (12).

3. CONCLUSIONS

The authors consider that their contribution is important because:

1. It synthesizes the a.r.a.s. method and system optimal control;
2. The approached application belongs to hydro aerodynamics (ballistics) as a critical case, one (characteristic) root being null $(r_3 = 0)$;
3. This dynamic system was optimal controlled, the optimizing parameter being the minimum stabilization time, and using a relay-type (on-off) regulator.

REFERENCES

1. Lupu, M., Florea, Ol., Criteria and applications regarding the absolute stability for the ships autopilot route adjustment, *Proceedings of the 13th International Conference "AFASES-Scientific Research and Education in the Air Force", Braşov, May 26-28, 2011, pp.*

pp.546-556, (ISSN, ISSN-L: 2247-3173), (2011).

2. Lupu, M., Florea, Ol., Lupu, C., Studies and Applications of Absolute Stability of the Nonlinear Dynamical Systems. *Annals of the Academy of Romanian Scientists*, Serie Science and Technology, vol. 2, 2(2013), pp. 183-205.
3. Lupu, M., Radu, Gh., Constantinescu, C.G., Airplanes or Rackets Flight Stabilization Optimal Control in Case of Rolling Perturbation, *Review of the Air Force Academy, The Scientific Informative Review, Vol. XIII, No. 2(29)/2015* (to be appear).
4. Lurie, A.Y., *Nonlinear problems from the automatic control.* Moskow: Ed. Gostehizdat, in Russian, (1951).
5. Merkin, D.R., *Introduction in the movement stability theory.* Moskow: Ed. Nauka, in Russian, (1987).
6. Nalepin, R.A. & Co., *Exact methods in the nonlinear system control in case of automatic regulation.* Moskow: Ed. M.S. (2004).
7. Pontreaguine, L. & Co., *Theorie mathematique des processus optimeaux.* Moscou: Editions Mir (1974).
8. Popov, V.M., *The hyperstability of automatic systems.* Bucharest: Ed. Academiei Romane (1966). Moskow: Ed. Nauka (1970). Berlin: Springer Verlag (1973).
9. Rouche, N., Habets, P., Leloy, M., *Stability theory by Liapunov's direct Method.* Springer Verlag (1977