ABOUT THE BEST APPROXIMATION OF CONTINUOUS FUNCTIONS BY POLYNOMIALS

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Abstract: *This paper presents some preliminary notions connected to the existence and the uniqueness of the best approximation polynomials.*

The following are discussed: limits, function oscillation and the distance between two functions.

The polynomials of Tchebychef are mentioned for continuous functions, their property of admitting a single polynomial of the best n *degree approximation.*

Keywords:.*approximation, polynomials, functions, continuity, boundedness*

1. BOUNDED FUNCTIONS. THE OSCILLATION OF A FUNCTION

In the following, we will consider the real functions $f(x)$, of a real *x* random, uniform and defined in a finite and closed interval (a,b), $a < b$.

A function $f(x)$ has a *superior limit* if there is an A number so that all the values considered by the function to the smaller than A. Contrarily, the function *does not have a superior limit*.

Let’s mark with $M(f)$ the superior limit or the maximum of the function $f(x)$. The definition of this number is:

If $f(x)$ is not superior limited, $M(f)$ equals $+\infty$.

If $f(x)$ is superior limited, $M(f)$ is defined by the condition that whatever is the positive number ε *, there is at least a point x for which*

$f(x) > M(f) - \varepsilon$

And whatever *x* is

$f(x) \le M(f)$

It is now clear what we have to understand by an *inferior limited* function and by a function that is *not inferior limited*. The definition of the *inferior limit* or that of the minimum $m(f)$ of the function $f(x)$ is analogue with that of the maximum $M(f)$.

A function that is limited both superior and inferior is a *limited function.* The difference $M(f) - m(f)$ is called the oscillation of the function $f(x)$ *in the* (a,b) *interval.*

2. CONTINUOUS FUNCTIONS

A continuous function on the (a,b) interval is known and understood. A continuous function is limited. A continuous function on an interval is *uniformly continuous* in this interval. This means that, being given a positive number ε we can determine another positive number δ in order to have $f(x') - f(x'') < \varepsilon$

Whatever the points x' and x'' checking the condition
$|x' - x''| < \delta$

A continuous function reaches the maximum $M(f)$ *and the minimum* $m(f)$ *of it.* Therefore, there is at least a point x'' so that $f(x'') = m(f)$. Moreover, we can state that $M(f)$ is also the *superior limit* of the function $f(x)$. In other words, $M(f)$ has the property that, whatever the positive number ε , there is an infinity of *x* points, so that
$f(x) > M(f) - \varepsilon$

And *at least one limited number of x points* so that
$f(x) > M(f) + \varepsilon$

The same, the minimum $m(f)$ coincides with *the inferior limit of the function* $f(x)$, this inferior limit having an analogue definition with the superior one.

The above mentioned extend immediately at functions that have more finite independent variables in a limited and closed domain.

Along these lessons we will need other properties that will be mentioned at the right time.

3. THE DISTANCE BETWEEN TWO FUNCTIONS

$f_1(x)$ and $f_2(x)$ being two functions, the number $M(|f_1 - f_2|)$ can be called their *distance*. If one of the functions is limited, and the other is not, their distance is boundless. If none of the functions is limited, their distance can be finite. If one of the functions is limited and their distance is finite, then the other function must be limited as well. The distance has the following properties, easily demonstrated:

a. $M(|f_1 - f_2|)$ is a positive or null number
b. $M(|f_1 - f_2|) = 0$ only if $f_1(x) \equiv f_2(x)$
c. $M(|Cf|) = CM(|f|)$, C being a positive constant
d. $M(|f_1 - f_2|) \le M(|f_1 - f_3|) + M(|f_2 - f_3|)$.

The best approximation problem, stated and studied below considers this distance definition.

4. THE BEST APPROXIMATION PROBLEM BY POLYNOMIALS

Let's study the problem. Let's consider the family or the *array* of the polynomials
$P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$

of *n* degree. A polynomial of the array is completely determined by the $a_0, a_1, ,..., a_n$ factors that are some negative, null or positive numbers. Therefore, any polynomial of *n* degree is also of $m > n$ degree. In other words, the array of polynomials of *n* degree contains the array of polynomials of a degree smaller than *n*.

Considering a function $f(x)$ we will say, by definition, that the distance $M(|f - P|)$ between this function and a $P(x)$ polynomial is the error or the *approximation* which $P(x)$ represents the $f(x)$ function.

For all polynomials of *n* degree, $M(|f - P|)$ has an inferior limit $\mu_n(f)$ or more simple μ_n. μ_n is by definition, *the best approximation* of the function $f(x)$ by *n* degree polynomials.

The problem of the best approximation by polynomials is the following:

Being given a function $f(x)$ *, determine the n degree polynomials for which* $M(|f-P|)$ *reaches its inferior limit* μ_n *and study this number* μ_n.

A $P(x)$ polynomial of *n* degree for which μ_n is reached will be called *a polynomial of the best approximation of n degree of the function* $f(x)$. Shortly, we will say that such a polynomial is a T_n polynomial and we will mark it with $T_n(x;f)$, $T_n(x)$ or T_n.

The problem of the best approximation polynomials has been first raised by the Russian mathematician P. L. Tchebychef. The results have been mentioned and completed by P. Kirchberger, E. Borel, L. Tonelli, Ch. De la Vallee Poussin.

5. DETERMINING THE T_N POLYNOMIALS IN SIMPLE CASES

The problem of the best approximation is not raised for a function that is not bounded, $M(|f-P|)$ being permanently equal with $+\infty$, because a polynomial is evidently a bounded function (in the interval (a,b)).

If $f(x)$ is a *n* degree polynomial, the best approximation is equal to zero because the function is itself a T_n polynomial. The reciprocal is true.

If we know the T_n polynomials for the $f(x)$ functions, then we know the T_n polynomials corresponding to the functions $f(x)+Q(x)$ and $Cf(x)$, where $Q(x)$ is a *n* degree polynomial and C a constant. Indeed, we have

$$M(|f-P|)=M(|f+Q-(P+Q|)=\mu_n(f)$$

And if $R(x)$ is a *n* degree polynomial,

$$M(|(f+Q)-R|)=M(|f-(R-Q)|)\geq \mu_n(f).$$

Therefore, $P(x)+Q(x)$ is a T_n polynomial for the function $f(x)+Q(x)$ and that any T_n polynomial corresponding to this function is of $P(x)+Q(x)$ form. We have

$$\mu_n(f+Q)=\mu_n(f)$$

Together with the relations

$$|C|M(|f-P|)=M(|Cf-CP|)=|C|\mu_n(f),$$

$$M(|Cf-R|)=M\left(\left|Cf-C\frac{R}{C}\right|\right)=|C|M\left(\left|f-\frac{R}{C}\right|\right)\geq |C|\mu_n(f)$$

Then $CP(x)$ is a T_n polynomial for the $Cf(x)$ functions and that any T_n polynomial corresponding this functions is of $CP(x)$ form. We have

$$\mu_n(Cf)=|C|\mu_n(f)$$

6. PRELIMINARY LEMMA

Let’s suppose that certain $P(x)$ polynomials of *n* degree have

$$P(x)<A \text{ in } (a,b). \tag{1}$$

We want to show that a_r factors are limited. For this, we take n+*1* distinct points $x_1,x_2,...,x_{n+1}$, situated in the interval (a,b) and consider the system

$$a_0x_r^n+a_1x_r^{n-1}+...+a_n=P(x_r),\ r=1,2,...,n+1.$$

The determinant of this system is different than zero, being no more than the Van Der Monde determinant of the numbers $x_1, x_2,..., x_{n+1}$. By pulling out the values of $a_0, a_1,,..., a_n$ with the help of the Cramer rule and regarding the relation (1) we state the following *preliminary lemma*:

If a $P(x)$ polynomial of n degree remains limited by a number A, in the (a,b) interval, then the factors $a_0, a_1,,..., a_n$ remain limited by a number λA, where λ depends only on n and on the (a,b) interval.

The value of λ can be specified. What is really important for us is that this number does not depend on the considered $P(x)$ polynomial.

The property remains evidently true as well for the case when the polynomials would be considered only on any linear and limited array having at least $n+a$ distinct points.

7. THE CONTINUITY OF $M(|f-P|)$.

The maxim $M(|f-P|)$ is definitely reached only if the function $f(x)$ is continuous.

Take ε an arbitrary positive number and write

$$A = M(|x|^n + |x|^{n-1} + ... + 1).$$

Let's suppose that

$$|a_r - a'_r| < \frac{\varepsilon}{A}, \qquad r = 0,1,2,..., n$$

Considering

$$P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_n$$

$$P_1(x) = a'_0 x^n + a'_1 x^{n-1} + ... + a'_n,$$

We have

$$M(|P - P_1|) \le [\max(|a_r - a'_r|)] M(|x|^n + |x|^{n-1} + ... + 1)$$

Where we usually write with $\max(c_1, c_2,..., c_k)$ or $\max\limits_{r=1,2,...,k}(c_r)$, or simpler, with $\max(c_r)$, on the biggest among the numbers $c_1, c_2,..., c_k$. We will use an analogue notation to indicate the smallest number between c_r.

We can then write

$$M(|P - P_1|) < \varepsilon$$

Here we deduce that

$$M(|f-P|) \le M(|f-P_1|) + M(|P-P_1|) < M(|f-P_1|) + \varepsilon$$

$$M(|f-P_1|) \le M(|f-P|) + M(|P-P_1|) < M(|f-P|) + \varepsilon$$

Consequently

$$|M(|f-P|) - M(|f-P_1|)| < \varepsilon$$

Which shows that:

$f(x)$ *being a limited function,* $M(|f-P|)$ is a continuous function of $a_0, a_1,,..., a_n$ coefficients.

Therefore, the inferior limit μ_n coincides with the inferior limit of the numbers $M(|f-P|)$.

8. THE EXISTENCE OF THE BEST APPROXIMATION POLYNOMIALS

We now examine the problem of the existence of the T_n polynomials. From the following, we can see that there are an unlimited number of *n* degree polynomials.

$$P_1(x), P_2(x), ..., P_m(x), ... \quad (2)$$

So that

$$M(|f - P_m|) \to \mu_n \ , \ m \to \infty$$

But the existence of a polynomial for which μ_n to be reached does not result yet, that is a polynomial $P(x)$ so that $M(|f - P|) = \mu_n$.

This does not come to surprise us. It is true that $M(|f - P|)$ is a continuous function compared to the *P* polynomial coefficients, but the variation domain of these factors is boundless and open.

If $M(|f|) = \mu_n$ then the zero polynomial is a T_n polynomial. In this case, the existence of at least a polynomial of the best approximation is proven.

Let's suppose the contrary case, that is $M(|f|) > \mu_n$. It is enough to only consider the *P* polynomials so that

$$M(|f - P|) < M(|f|)$$

There is an infinite number of such polynomials of *n* degree.

But,

$$M(|P|) \le M(|f - P|) + M(|f|)$$

$$M(|P|) < 2M(|f|) \quad (3)$$

In other words, we can suppose that the (2) polynomials are chosen in such a way that they check the equality (3).

If we write

$$P_m = a_0^{(m)} x^n + a_1^{(m)} x^{n-1} + ... + a_n^{(m)}, \ m = 1,2,...$$

We know that there is a *B* number, that only depends on $M(|f|)$, $[B = 2\lambda M(|f|)]$, so that

$$\left|a_r^{(m)}\right| < B, \ r = 0,1,2,...n \, ; \ m = 1,2,...$$

From the array

$$a_0^{(1)}, a_0^{(2)}, ..., a_0^{(m)}, ...$$

Which is limited, we can extract a partial array to have a a_0^* finite limit

$$a_0^{(k_1)}, a_0^{(k_{12})}, ..., a_0^{(k_{1m})}, ... \to a_0^* \quad (4)$$

We then consider the array

$$a_1^{(k_1)}, a_1^{(k_{12})}, ..., a_1^{(k_{1m})}, ...$$

From this array we can extract a partial array to have a a_1^* finite limit

$$a_1^{(k_1)}, a_1^{(k_2)}, a_1^{(k_{23})}, ..., a_1^{(k_{2m})}, ... \to a_1^*$$

We will also have

$$a_0^{(k_1)}, a_0^{(k_2)}, a_0^{(k_{23})} ..., a_0^{(k_{2m})}, ... \to a_0^*$$

Because this array is extracted from (4).

By repeating this procedure de *n+1* times, we see that, in the end, from the (2) polynomial array we can extract a partial array

$$P_{k_1}, P_{k_2}, ..., P_{k_m}, ...$$

So that

$a_r^{(k_1)}, a_r^{(k_2)}, a_r^{(k_3)} ..., a_{r0}^{(k_m)}, ... \quad r = 0,1,2..., n$

a_r^* being finite numbers

If we now write

$P^*(x) = a_0^* x^n + a_1^* x^{n-1} + ... + a_n^*,$

We see that

$$M(|f - P^*|) = \mu_n \tag{5}$$

The polynomial $P^*(x)$ for which we have the equality (5) is therefore a best approximation polynomial of *n* degree of the function $f(x)$. Therefore, we can state the following property:

Any limited function $f(x)$ admits at least one polynomial of the best n degree approximation.

The necessary and adequate condition for μ_n to be zero is that $f(x)$ is an n degree polynomial.

We know that this condition is adequate. Its necessity results from the fact that there is a $P(x)$ polynomial so that $M(|f - P|) = 0$, where $f(x) \equiv P(x)$. In the case when $f(x)$ is not an *n* degree polynomial, μ_n is a positive number.

9. THE POLYNOMIALS OF TCHEBYCHEF FOR A CONTINUOUS FUNCTION

We will now suppose that the $f(x)$ function is continuous and take $T_n(x)$ a polynomial of the best approximation of *n* degree. The result of $f(x) - T_n(x)$ will necessarily reach one of the values $\pm \mu_n$. We state the number of points in which these values are reached.

Let's suppose that

$f(x_r) - T_n(x_r) = \pm \mu_n, \quad r = 1,2,..., m$

where $x_1, x_2, ..., x_m$ are *m* distinct points and $m \le n+1$. In all the other points of the interval (a,b), we have $|f - T_n| < \mu_n$.

Take $Q(x)$ the Lagrange polynomial conditioned by

$Q(x_r) = f(x_r) - T_n(x_r) = \pm \mu_n, \quad r = 1,2,..., m$

The LAGRANGE polynomial, given by the interpolation formula of LAGRANGE, is the *smallest degree* polynomial taking the values $A_1, A_2, ..., A_k$. This polynomial is *unique* and its degree is at most equal with $k-1$.

The $Q(x)$ polynomial is therefore of *n* degree.

We close every point x_r in an I_r interval, having as a centre the point x_r and as length δ_r.

Being given a positive number $\varepsilon < \mu_n$, we can select a positive number δ together with the lengths δ_r so that:

1^0. Taking $\delta_r \le \delta$, the intervals $I_1, I_2, ..., I_m$ should not have any common point.

2^0. The oscillation of the functions $f(x) - T_n(x)$ and $Q(x)$ should be smaller than ε in any length interval $\le \delta$.

Hence it appears that in an I_r interval, the $f - T_n$ and *Q* functions cannot be cancelled, therefore, they keep a constant sign (the same sign)

Let's suppose that x_r is a point in which $f(x_r) - T_n(x_r) = Q(x_r) = \mu_n$, then in the I_r interval, we have

$$\mu_n - \varepsilon < f - T_n \leq \mu_n$$

$$\mu_n - \varepsilon < Q \leq \mu_n + \varepsilon .$$

Let's select the λ positive number so that

$$\lambda < \frac{\mu_n - \varepsilon}{\mu_n + \varepsilon} \tag{6}$$

Then in the I_r interval, we have

$$0 < \mu_n - \varepsilon - \lambda(\mu_n + \varepsilon) < f - T_n - \lambda Q < \mu_n - \lambda(\mu_n - \varepsilon)$$

In an x_r point where $f(x_r) - T_n(x_r) = Q(x_r) = -\mu_n$, we have

$$-\mu_n \leq f - T_n < -\mu_n + \varepsilon$$

$$-\mu_n - \varepsilon < Q < -\mu_n + \varepsilon$$

And in addition to the same condition (6) we deduce

$$-\mu_n + \lambda(\mu_n - \varepsilon) \leq f - T_n - \lambda Q < -\mu_n + \varepsilon + \lambda(\mu_n + \varepsilon) < 0$$

It appears that in the I_r intervals

$$\left| f - T_n - \lambda Q \right| < \mu_n - \lambda(\mu_n - \varepsilon) < \mu_n$$

From our initial hypothesis it also appears that in all points of the closed domain, obtained from the (a,b) interval, by taking out the I_r intervals, we have

$$\left| f - T_n \right| \leq \mu' < \mu_n ,$$

Where μ' is a fixed number

If we take a small enough λ so that

$$\lambda < \frac{\mu_n - \mu'}{2M(|Q|)} \tag{7}$$

We will also have

$$|\lambda Q| < \frac{\mu_n - \mu'}{2},$$

$$\left| f - T_n - \lambda Q \right| \leq \left| f - T_n \right| + |\lambda Q| < \mu' + \frac{\mu_n - \mu'}{2} = \frac{\mu_n + \mu'}{2} < \mu_n$$

Besides the I_r intervals and in the extremities of these intervals.

Hence it results that in the entire (a,b) interval,

$$\left| f - T_n - \lambda Q \right| < \mu_n$$

Therefore, if λ checks the inequalities (6) and (7), the $T_n + \lambda Q$ polynomial renders a better approximation, contrarily to the hypothesis, and so we have the following property:

The result of $f(x) - T_n(x)$ *reaches the values* $\pm \mu_n$ *in at least n+2 points*

10. THE COMPLETION OF THE PREVIOUS RESULT

We can complete the previous result. The result of $f(x) - T_n(x)$ must reach both $+\mu_n$ and $-\mu_n$ values. Let's suppose, for example that $+\mu_n$ would not be reached, we then have

$$-\mu_n \leq f - T_n \leq \mu' < \mu_n$$

μ' being a fixed number. By taking a positive constant λ we will have

$$-\mu_n + \lambda \leq f - T_n + \lambda \leq \mu' + \lambda$$

Therefore, if we take $\lambda < \mu_n - \mu'$, we have everywhere
$|f - T_n + \lambda| < \mu_n$.

The $T_n - \lambda$ polynomial gives a better approximation, which is contrary to the hypothesis.

Moreover, we can state the number of the points where μ_n and the number of the $-\mu_n$ points is reached. Let's suppose, for example that
$f(x_r) - T_n(x_r) = \mu_n, \quad r = 1,2,..., m.$

In all the other points having
$-\mu_n \le f - T_n < \mu_n$.

Take the intervals I_r having the centre in x_r and a small enough δ_r length for the intervals I_r to have no common point. Take x'_r, x''_r as extremities of the I_r interval and then form the polynomial
$Q(x) = (x - x'_1)(x - x''_1)(x - x'_2)(x - x''_2)...(x - x'_m)(x - x''_m)$

We have $Q(x) < 0$ in the open intervals I_r and $Q(x) > 0$ besides I_r closed intervals.
$\mu' \le f - T_n < \mu_n$

μ' being a positive number $< \mu_n$.

If the positive number λ checks the inequality

$$\lambda < \frac{\mu'}{M(|Q|)} \tag{8}$$

Then we have in the I_r intervals
$0 < \mu' + \lambda Q \le f - T_n + \lambda Q \le \mu_n + \lambda Q < \mu_n$

The last inequality is justified because we could not have an equality except in a point where we have in the same time $f - T = \mu_n$ și $Q=0$.

In the entire closed domain, obtained from *(a,b)* by taking out the I_r intervals, we have
$-\mu_n \le f - T_n \le \mu'' < \mu_n$,

μ'' being a fixed number.

Taking λ so that

$$\lambda < \frac{\mu_n - \mu''}{2M(|Q|)} \tag{9}$$

We have in this domain

$$-\mu_n \le -\mu_n + \lambda Q \le f - T_n - \lambda Q \le \mu'' + \frac{\mu_n - \mu''}{2} = \frac{\mu_n + \mu''}{2} < \mu_n .$$

The first inequality is explained exactly as above.

Therefore, if λ verifies the inequalities (8),(9) we have in the entire *(a,b)* interval
$|f - T_n + \lambda Q| < \mu_n$

And we can see that the $T_n - \lambda Q$ polynomial gives a better approximation than μ_n.

The *Q(x)* polynomial is of *2m* degree, so we reach an inconsistence $2m \le n$.

If x_r would be points where $-\mu_n$ is reached, we can bring absolutely analogue arguments, therefore, we can state the following property:

The result between $f(x) - T_n(x)$ reaches in at least $\left[\frac{n+2}{2}\right]$ points the value μ_n and in at least $\left[\frac{n+2}{2}\right]$ points the value $-\mu_n$. $[\alpha]$ stands for the number of integers covered in α.

11. ABOUT THE ARRAY OF THE T_n POLYNOMIALS.

Let's assume that the $f(x)$ function takes two T_n distinct polynomials. If P, P_1 are these two polynomials, then we have

$$M(|f-P|)=M(|f-P_1|)=\mu_n$$

If α,β are two positive numbers, we have

$$\begin{aligned}\mu_n &\le M\left(\left|f-\frac{\alpha P+\beta P_1}{\alpha+\beta}\right|\right)=M\left(\left|\frac{\alpha(f-P)}{\alpha+\beta}+\frac{\beta(f-P_1)}{\alpha+\beta}\right|\right)\le \\ &\le \frac{\alpha M(|f-P|)+\beta M(|f-P_1|)}{\alpha+\beta}=\mu_n\end{aligned} \tag{10}$$

$$M\left(\left|f-\frac{\alpha P+\beta P_1}{\alpha+\beta}\right|\right)=\mu_n$$

Hence it results that the polynomial $\frac{\alpha P+\beta P_1}{\alpha+\beta}$ is as well a T_n polynomial, therefore:

If a limited function admits two T_n distinct polynomials, then it admits infinity (countless) of such polynomials.

Each $P(x)=a_0x^n+a_1x^{n-1}+...+a_n$ polynomial can have point A $a_0,a_1,...,a_n$ coordinates in the ordinary space with *n+ 1* dimension. We can then see that:

Points A corresponding to the T_n polynomials of a limited function, form a convex, closed and limited domain.

If the T_n polynomial is unique, then this domain is reduced to a single point.

If the *(a,b)* interval is symmetrical compared to the origin, $a=-b$, and if $f(x)$ is an even function, $f(x)=f(-x)$, there is a T_n polynomial which is even a well. Indeed, we can immediately see that $T_n(-x)$ is a T_n polynomial as well. In the same manner, $\frac{T_n(x)+T_n(-x)}{2}$, is an even polynomial. In this case $\mu_{2n+1}(f)=\mu_{2n}(f)$

If the function is odd, $f(-x)=-f(x)$, there is a T_n polynomial which is odd as well. In this case $\mu_{2n}(f)=\mu_{2n-1}(f)$.

12. THE UNIQUENESS OF THEBYCHEF POLYNOMIALS

In the case of continuous functions, the preceding ones allow us to draw an important conclusion.

If P, P_1 are two T_n distinct polynomials, the $P_2=\frac{P+P_1}{2}$ polynomial is a T_n polynomial as well. The inequality (10) shows us that in a x' point where we have $f(x')-P_2(x')=\pm\mu_n$, we also have to have

$$f(x')-P(x')=f(x')-P_1(x')=\pm\mu_n$$

$$P(x')=P_1(x')$$

In the virtue of the above properties, the P, P_1 polynomials coincide in at least *n+2* points, they are therefore, identical. Hence it results the property:

A continuous $f(x)$ function admits a single polynomial of the best n degree approximation.

The uniqueness results only from the fact that $|f - T_n|$ reaches its maxim in at least *n+ 1 points*. Indeed, two *n* degree polynomials which have the same value in *n+ 1 point*, have the same value everywhere.

If the interval *(a,b)* is symmetrical compared to the origin and $f(x)$ is an even function, the $T_n(x)$ polynomial is even as well, therefore, $T_{2n+1} = T_{2n}$. If the function is odd, the $T_n(x)$ polynomial is odd as well, and $T_{2n} = T_{2n-1}$.

If the function is not continuous, the T_n polynomial is not generally unique. We can observe that T_0 is always unique and equal with $\frac{M(f)+m(f)}{2}$. Take the function

$$f = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \end{cases}$$

It can be seen that we need to have $\mu_n \ge 1$.But the zero polynomial approximates 1, therefore $\mu_n = 1$ whatever *n*. All T_n polynomials must be null in origin. The Cx polynomials where *C* is a constant, there are T_n polynomials for $0 \le C \le 2$ and whatever $n > 0$ is.

REFERENCES

[1] T. Popoviciu, *Sur l'aproximation des functions convexes d'ordre superieur,* Mathematica (Cluj) 1935;

[2] L. Lupaş, A. Lupaş, *Polynomials of binomial type and approximation operators,* Studia Universitatea Babeş-Bolyai Mathematica (1987)

[3] D.D. Stancu, *Approximation of functions by a new class of linear polynomial operators,* Revista Roumanie Math Pures et Application (1968)